

ON NONISOTHERMAL TWO-FLUID CHANNEL FLOWS

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Abstract. Two-fluid channel flows arise in different kinds of coating technologies. The corresponding mathematical models represent two-dimensional free boundary value problems for the Navier-Stokes equations or their modifications. In this paper we are concerned with the so-called Boussinesq-approximation of the coupled heat- and mass transfer. Thermocapillary convection is included. The solvability of two related stationary problems is discussed. The solution techniques of both problems are quite different. The obtained results generalize previous results for similar isothermal problems.

Key words: Free boundary value problems, viscous nonisothermal channel flows, two-fluid flows, Boussinesq-approximation, thermocapillary convection

1. Introduction

Under thermocapillary convection one understands a fluid motion driven by surface-tension gradients on a liquid-liquid interface, where these gradients arise from surface-temperature gradients and the temperature dependence of surface tension. This type of convection is very important in many technological and scientific applications; interesting examples may be found in the field of materials processing, particularly in coating and solidification processes or in crystal-growth processes (cf. [2, 4, 14, 16, 18]).

In this article we consider two problems for plane stationary flows with two viscous incompressible heat-conducting fluids in each (having kinematic viscosities $\nu_i > 0$, densities $\rho_i > 0$ and thermal conductivities λ_i , $i = 1, 2$) through different **horizontal** channels. Emphasize that the corresponding problems will be formulated in dimensionless form. The concrete transition to that formulation can be found in [19].

Let us formulate the first problem which we will denote by Problem (I) in the sequel. We consider the two-fluid flow within a perturbed horizon-

tal channel of width 1 between the walls S_0 and S_2 (cf. Fig. 1). The moving bottom S_0 of it is given by the formula $S_0 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 = 0, -\infty < x_1 < +\infty\}$ and the fixed top S_2 has the representation $S_2 = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = 1 + \varepsilon \varphi_2(x_1), -\infty < x_1 < +\infty\}$. Furthermore, we suppose that φ_2 has a compact support. Since the channel is horizontal, the direction \mathbf{e}_g of gravitational force is equal to $(0, -1)^T$ (cf. Fig. 1).

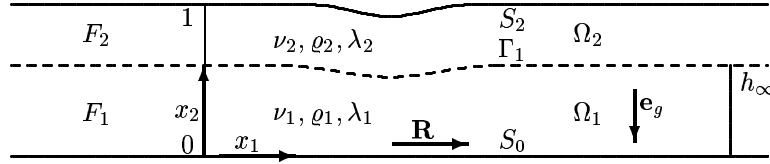


Figure 1. Flow domain for Problem (I).

We study the plane stationary flow of two viscous incompressible heat-conducting fluids generated by a pressure gradient downstream in the perturbed channel, by a temperature gradient in the transverse direction and by motion of the lower wall S_0 with constant velocity $\mathbf{R} = (R, 0)^T$. This means mathematically that the volume flux F_i in each fluid layer Ω_i ($i = 1, 2$) is prescribed. Suppose that the free interface Γ_1 separating two fluid layers admits the parametrization $\Gamma_1 = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \psi_1(x_1), -\infty < x_1 < +\infty\}$, where the function ψ_1 is a priori unknown and has to be found. Emphasize that F_i ($i = 1, 2$), R are not necessarily positive for this channel flow.

Let h_∞ be the constant limit of $\psi_1(x_1)$ at infinity. Obviously, it should hold $0 < h_\infty < 1$. Problem (I) has the following form: to find a vector of velocity $\mathbf{v} = (v_1(x_1, x_2), v_2(x_1, x_2))^T$, a pressure $p(x_1, x_2)$, a temperature $\theta(x_1, x_2)$ and a function $\psi_1(x_1)$ satisfying in the domain $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_2 < \psi_1(x_1), -\infty < x_1 < +\infty\}$ and $\Omega_2 = \{\mathbf{x} \in \mathbb{R}^2 : \psi_1(x_1) < x_2 < 1 + \varepsilon \varphi_2(x_1), -\infty < x_1 < +\infty\}$ the Boussinesq-approximation of the coupled heat- and mass transfer (cf. [3])

$$\begin{cases} (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \nabla^2 \mathbf{v} + \frac{1}{\rho} \nabla p = (g - \gamma \theta) \mathbf{e}_g, \\ \nabla \cdot \mathbf{v} = 0, \\ (\mathbf{v} \cdot \nabla) \theta - \lambda \nabla^2 \theta = 0, \end{cases} \quad (1.1)$$

and the boundary and integral conditions

$$\begin{aligned} \mathbf{v}|_{S_0} &= (R, 0)^T, & \mathbf{v}|_{S_2} &= \mathbf{0}, \\ \theta|_{S_0} &= \theta_0, & \theta|_{S_2} &= \theta_2, \end{aligned}$$

$$\begin{cases} [\theta]|_{\Gamma_1} = 0, & \left[\lambda \frac{\partial \theta}{\partial n} \right] \Big|_{\Gamma_1} = 0, & [\mathbf{v}]|_{\Gamma_1} = \mathbf{0}, \\ \mathbf{v} \cdot \mathbf{n}|_{\Gamma_1^-} = 0, & [\mathbf{t} \cdot \mathbf{S}(\mathbf{v}) \mathbf{n}]|_{\Gamma_1} = -b \frac{\partial \theta}{\partial t} \Big|_{\Gamma_1^-}, \\ \frac{d}{dx_1} \frac{\psi_1'(x_1)}{\sqrt{1 + \psi_1'(x_1)^2}} = \frac{1}{\sigma(\theta)} [-p + \mathbf{n} \cdot \mathbf{S}(\mathbf{v}) \mathbf{n}]|_{\Gamma_1}, \\ \lim_{|x_1| \rightarrow +\infty} \psi_1(x_1) = h_\infty, & \int_{\delta_1(\hat{q})} v_1 dx_2 = F_1, \\ & \int_{\delta_2(\hat{q})} v_1(\hat{q}, x_2) dx_2 = F_2. \end{cases} \quad (1.2)$$

In [2] it was shown that for a large number of liquids the surface tension σ can be regarded as a linear function of the temperature θ along the free interface Γ_1 (cf. also [14, 18])

$$\sigma(\theta) = a - b\theta. \quad (a, b > 0) \quad (1.3)$$

In Problem (I) the symbol $\delta_i(\hat{q})$ denotes the intersection of Ω_i with the vertical line $x_1 = \hat{q}$. By γ_m we denote the thermal expansion coefficient of the m -th fluid ($m = 1, 2$). The symbol g means the acceleration of gravity. The values θ_0 and θ_2 are the (constant) given temperatures of the walls S_0 and S_2 , respectively. Without loss of generality one can suppose that $\theta_0 = 0$ and that θ is in fact the difference between the physical temperature and θ_0 .

Furthermore, the following notations have been used: \mathbf{n} and \mathbf{t} are unit vectors normal and tangential to Γ_1 and oriented as x_2, x_1 , respectively. By $\mathbf{a} \cdot \mathbf{b}$ we mean the inner product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2)^T$ is the gradient operator, $\nabla p = \text{grad } p, \nabla \cdot \mathbf{v} = \text{div } \mathbf{v}$, $\varrho|_{\Omega_m} = \varrho_m$ ($m = 1, 2$) is the restriction of ϱ to Ω_m (analogously for ν and λ). ∇^2 denotes the Laplace operator. By $\mathbf{S}(\mathbf{v})$ we denote the deviatoric stress tensor, i.e. a matrix with elements $S_{ij}(\mathbf{v}) = \varrho\nu(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ ($i, j = 1, 2$). The symbol $[w]|_{\Gamma_1}$ expresses the jump of w crossing the free interface Γ_1 , i.e.

$$[w(\mathbf{x}_0)]|_{\Gamma_1} := \lim_{\mathbf{y} \rightarrow \mathbf{x}_0} w(\mathbf{y}) - \lim_{\mathbf{x} \rightarrow \mathbf{x}_0} w(\mathbf{x}), \quad (\mathbf{x}_0 \in \Gamma_1, \mathbf{y} \in \Omega_1, \mathbf{x} \in \Omega_2),$$

and the symbol $w|_{\Gamma_1^-}$ denotes the limit from below at the interface Γ_1 , more precisely

$$w(\mathbf{x}_0)|_{\Gamma_1^-} := \lim_{\mathbf{y} \rightarrow \mathbf{x}_0} w(\mathbf{y}), \quad (\mathbf{x}_0 \in \Gamma_1, \mathbf{y} \in \Omega_1). \quad (1.4)$$

Note that the left-hand side of (1.2)₆ (i.e. of the sixth equation in (1.2)) is equal to the curvature $K(x_1)$ of Γ_1 .

The second flow under consideration is also steady-state and has some features of a slot coating process. The channel is again horizontal, unbounded in both directions and contains a semi-infinite inner wall (cf. Fig.2). The lower wall $S_0 := \{\mathbf{x} \in \mathbb{R}^2 : -\infty < x_1 < +\infty, x_2 = 0\}$ is again moving with constant

velocity $\mathbf{R} = (R, 0)^T$ ($R \geq 0$). The upper wall (which is a straight line in this case) $S_2 := \{\mathbf{x} \in \mathbb{R}^2 : -\infty < x_1 < +\infty, x_2 = 1\}$ is at rest. Furthermore, we are given the partial inner wall $S_3 := \{\mathbf{x} \in \mathbb{R}^2 : -\infty < x_1 < 0, x_2 = h_1$ ($0 < h_1 < 1$)}. Thus, in fact we have two separated parallel channels for negative values of x_1 . Both viscous fluids are flowing out of the two channels and behind the point $Q(0, h_1)$ they are joining and creating a free interface $\Gamma_1 := \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_1 < +\infty, x_2 = \psi_1(x_1)\}$ where ψ_1 is unknown a priori and has to be found. It is supposed that the free interface Γ_1 separates from the inner wall S_3 at its endpoint Q .

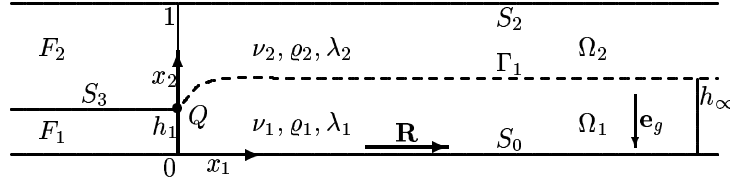


Figure 2. Flow domain for Problem (II).

By $\Omega_1 := \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_2 < h_1$ if $-\infty < x_1 \leq 0$ and $0 < x_2 < \psi_1(x_1)$ if $0 < x_1 < +\infty\}$ we denote the flow domain of the lower fluid. By Ω_2 , the flow domain of the upper fluid, respectively, we understand the set $\Omega_2 := \{\mathbf{x} \in \mathbb{R}^2 : h_1 < x_2 < 1$ if $-\infty < x_1 \leq 0$ and $\psi_1(x_1) < x_2 < 1$ if $0 < x_1 < +\infty\}$. Finally, by $\Omega := \Omega_1 \cup \Omega_2$ we mean the union of both fluid layers. The direction of the gravitational force is again the vector $\mathbf{e}_g = (0, -1)^T$. We study the two-fluid flow through the channel Ω caused by a pressure gradient downstream, by temperature gradients in the spanwise direction and by motion of the lower channel wall. This means mathematically that the positive volume flux F_i in each liquid layer Ω_i ($i = 1, 2$) is prescribed and the **final** fluid layer thicknesses h_∞ and $(1 - h_\infty)$ are to be determined.

An interpretation of such a flow could be the flow of two liquids coming from different reservoirs (i.e. slots or chambers) and flowing commonly in one channel after their unification. In slot coaters such flows occur on some parts of the coater. The corresponding motion as well as the final layer thicknesses are important in that case.

Let h_∞ be the constant limit of $\psi_1(x_1)$ as $x_1 \rightarrow +\infty$. Obviously, it holds $0 < h_\infty < 1$. Then Problem (II) has the following description: to find a vector of velocity \mathbf{v} , a pressure p , a temperature θ and a function ψ_1 satisfying in the domain Ω the Boussinesq-approximation of the coupled heat- and mass transfer

$$\begin{cases} (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \nabla^2 \mathbf{v} + \frac{1}{\rho} \nabla p = (g - \gamma \theta) \mathbf{e}_g, \\ \nabla \cdot \mathbf{v} = 0, \\ (\mathbf{v} \cdot \nabla) \theta - \lambda \nabla^2 \theta = 0, \end{cases} \quad (1.5)$$

and the boundary and integral conditions

$$\begin{aligned}
 \mathbf{v}|_{S_0} &= (R, 0)^T, & \mathbf{v}|_{S_2} &= \mathbf{0}, & \mathbf{v}|_{S_3^\pm} &= \mathbf{0}, \\
 \theta|_{S_0} &= \theta_0, & \theta|_{S_2} &= \theta_2, & \theta|_{S_3^\pm} &= \theta_3, \\
 \left\{ \begin{array}{l}
 [\theta]|_{\Gamma_1} = 0, & \left[\lambda \frac{\partial \theta}{\partial n} \right] \Big|_{\Gamma_1} = 0, & [\mathbf{v}]|_{\Gamma_1} = \mathbf{0}, \\
 \mathbf{v} \cdot \mathbf{n}|_{\Gamma_1^-} = 0, & [\mathbf{t} \cdot \mathbf{S}(\mathbf{v}) \mathbf{n}]|_{\Gamma_1} = -b \frac{\partial \theta}{\partial t} \Big|_{\Gamma_1^-}, \\
 \frac{d}{dx_1} \frac{\psi'_1(x_1)}{\sqrt{1 + \psi'_1(x_1)^2}} = \frac{1}{\sigma(\theta)} [-p + \mathbf{n} \cdot \mathbf{S}(\mathbf{v}) \mathbf{n}]|_{\Gamma_1}, \\
 \lim_{x_1 \rightarrow +\infty} \psi_1(x_1) = h_\infty, & \int_{\delta_1(\hat{q})} v_1 dx_2 = F_1, \\
 \int_{\delta_2(\hat{q})} v_1(\hat{q}, x_2) dx_2 = F_2.
 \end{array} \right. \quad (1.6)
 \end{aligned}$$

Note that surface tension is the same as in (1.3). For the one-side limits at S_3^\pm we use analogous symbols as in (1.4). The fluid layer thickness h_∞ has to be determined.

2. General Solution Techniques

Mathematical problems for the stationary flows of a viscous incompressible fluid with a free boundary were investigated by many authors. Numerous references on this topic can be found - e.g. - in the bibliographies of [9, 13, 22, 23]. In the analytical investigations [2, 4, 8, 14] and [18] the temperature dependence was additionally taken into account. Numerical studies of nonisothermal free boundary problems one can find in the papers [2] and [19]. Coating flows which usually include static or dynamic contact points were studied in [6, 5, 10, 17, 19, 24, 25]. In all papers containing either compact or semi-infinite free boundary value problems the same general solution scheme developed in [7, 15] has been used.

Let us shortly recall this scheme on Problem (II). The starting problem is divided into two problems: the boundary value problem for the differential equations (1.5) in a fixed domain and the problem of finding the free boundary Γ_1 from the equation

$$K(x_1) = \frac{1}{\sigma(\theta)} [-p(\mathbf{x}) + \mathbf{n} \cdot \mathbf{S}(\mathbf{v}) \mathbf{n}]|_{\Gamma_1} \quad (2.1)$$

and from the corresponding boundary conditions. The solution of the free boundary problem can be found by the method of successive approximations. At every step of successive approximations the system (1.5) is solved in a fixed domain. The obtained solution is substituted into the right-hand side of (2.1) and by solving this equation one obtains the next iteration for the free

boundary Γ_1 . Thus, one gets a new domain in which system (1.5) has to be solved again. So, this scheme can be illustrated by the diagram

$$\Gamma_1^0 \rightarrow \Omega^0 \rightarrow (\mathbf{v}^1, p^1, \theta^1) \rightarrow \Gamma_1^1 \rightarrow \Omega^1 \rightarrow (\mathbf{v}^2, p^2, \theta^2) \rightarrow \dots \quad (2.2)$$

Note that in this method at every step of successive approximations the construction of (\mathbf{v}, p, θ) is separated from the construction of the free boundary Γ_1 . On the other hand, for free boundary problems in which the unknown flow domain is unbounded in two directions as in Problem (I) the described scheme is not applicable (cf. [9, 13] and others).

In order to solve such problems in [9], and independently in [1], a different scheme was proposed which based on a linearization of the original problem on an appropriate exact solution in the unperturbed “uniform” flow domain, say $\Pi = \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_2 < 1\}$. The main difference of this scheme from the previous one is that on each step of iterations the determination of \mathbf{v}, p, θ is not separated from the determination of the free boundary Γ_1 (i.e. from the determination of the functions ψ_1 describing Γ_1).

In order to solve Problem (I), this second scheme should be applied. For Problem (I) this scheme can be illustrated by the diagram

$$(\mathbf{v}^0, p^0, \theta^0, \psi_1^0) \rightarrow (\mathbf{v}^1, p^1, \theta^1, \psi_1^1) \rightarrow \dots \rightarrow (\mathbf{v}^m, p^m, \theta^m, \psi_1^m) \rightarrow \dots$$

where on each step of iterations the linearized problem is solved in the same “uniform” domain and the functions \mathbf{v}, p, θ and ψ_i ($i = 1, 2$) are determined simultaneously.

A significant part in deriving the correct linearization takes the determination of exact solutions of the nonlinear problems in a “uniform” (not distorted) flow domain. These exact (basic) solutions in the uniform domain Π will be calculated in Section 5 - Appendix. They are also important for the numerical flow simulation: they can be used as inlet boundary data in more complicated problems. Furthermore, the basic solution represents an asymptotic solution as $x_1 \rightarrow +\infty$ in Problem (II). In [13] and [20] the analogous isothermal problems (without any inclusion of temperature) to problems (I) and (II), respectively, were solved by numerical methods. Note that the layer thicknesses h_∞ and $(1 - h_\infty)$ of the fluids and, therefore the whole exact solutions in the uniform domain, could be not unique (see Appendix). For the case of Navier-Stokes equations such nonunique solutions were found in [11] for the first time.

3. Function Spaces

When studying Problem (I) it is very convenient to work with weighted Sobolev spaces. Let Π_m ($m = 1, 2$) be the strip-like domains

$$\begin{aligned} \Pi_1 &:= \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_2 < h_\infty, -\infty < x_1 < +\infty\}, \\ \Pi_2 &:= \{\mathbf{x} \in \mathbb{R}^2 : h_\infty < x_2 < 1, -\infty < x_1 < +\infty\}, \end{aligned}$$

and $\Pi = \Pi_1 \cup \Pi_2$ their union, where $h_\infty \in (0, 1)$ is the root to equation (5.4) (or is one of the roots to (5.4)). We introduce the space $W_\beta^{l,2}(\Pi)$ of functions

\mathbf{u} on Π with restrictions $\mathbf{u}^{(m)} = \mathbf{u}|_{\Pi_m}$ belonging to $W_\beta^{l,2}(\Pi_m)$ ($m = 1, 2$) and having the finite norms

$$\|\mathbf{u}^{(m)}; W_\beta^{l,2}(\Pi_m)\| = \|\mathbf{u}^{(m)} \exp\left(\beta\sqrt{1+x_1^2}\right); W^{l,2}(\Pi_m)\|, \quad (m = 1, 2)$$

where $W^{l,2}(\Pi_m)$ is the usual Sobolev space. The norm in $W_\beta^{l,2}(\Pi)$ is given by

$$\|\mathbf{u}; W_\beta^{l,2}(\Pi)\| = \sum_{m=1}^2 \|\mathbf{u}^{(m)} \exp\left(\beta\sqrt{1+x_1^2}\right); W^{l,2}(\Pi_m)\|.$$

If $\beta > 0$, then elements of $W_\beta^{l,2}(\Pi)$ vanish exponentially as $|x_1| \rightarrow \infty$ and, if $\beta < 0$, then elements $\mathbf{u} \in W_\beta^{l,2}(\Pi)$ might exponentially increase as $|x_1| \rightarrow \infty$.

The spaces $W_\beta^{l-1/2,2}(\mathbb{R})$ of functions defined on \mathbb{R} can be introduced analogously. Let $S = \{x \in \Pi : x_1 \in \mathbb{R}, x_2 = h \in [0, 1]\}$ be a line. Denote by $W_\beta^{l-1/2,2}(S)$ the spaces of traces on S of functions from $W_\beta^{l,2}(\Pi)$. Then $W_\beta^{l-1/2,2}(\mathbb{R})$ coincides with $W_\beta^{l-1/2,2}(S)$, i.e. if $\mathbf{u} \in W_\beta^{l-1/2,2}(\Pi)$, then $\mathbf{u}(\cdot, h) \in W_\beta^{l-1/2,2}(\mathbb{R})$.

When investigating Problem (II) we are using weighted Hölder spaces. This is due to a better handling of static contact points.

Let B be an arbitrary domain in \mathbb{R}^2 and $N \subset \bar{B}$ a manifold of dimension less than 2. The symbol $\varrho_N(\mathbf{x})$ denotes (in this section only) the distance $\text{dist}(\mathbf{x}, N) := \inf_{\mathbf{y} \in N} |\mathbf{x} - \mathbf{y}|$. Let $\beta = (\beta_1, \beta_2)$ be a multiindex with

$$|\beta| = \beta_1 + \beta_2 \quad \text{and} \quad D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \quad (\beta_i \in \mathbb{N} \cup \{0\}).$$

The symbol $[r]$ will denote the integer part of r (only in this section). $C^r(B)$ ($r > 0$, non-integer) denotes the Hölder space of functions defined in a domain $B \subset \mathbb{R}^2$ with a finite norm

$$|u|_B^{(r)} = \sum_{|\beta| < r} \sup_{\mathbf{x} \in B} |D^\beta u| + \sum_{|\beta| = [r]} \sup_{\mathbf{x}, \mathbf{y} \in B} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{r-[r]}}.$$

Let $\hat{C}_s^r(B, N)$ be the weighted Hölder space of functions defined in $B \setminus N$ and having a finite norm

$$\begin{aligned} |u|_{\hat{C}_s^r(B, N)} &= \sum_{|\beta| < r} \sup_{\mathbf{x} \in B \setminus N} \varrho_N^{|\beta|-s}(\mathbf{x}) |D^\beta u(\mathbf{x})| \\ &+ \sum_{|\beta| = [r]} \sup_{\mathbf{x} \in B \setminus N} \varrho_N^{r-s}(\mathbf{x}) \sup_{|\mathbf{x}-\mathbf{y}| < \frac{1}{2}\varrho_N(\mathbf{x})} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{r-[r]}}. \end{aligned}$$

$C_s^r(B, N)$ ($r > s > 0$; r, s non-integer) denotes the space of functions with a finite norm

$$\begin{aligned}
|u|_{C_s^r(B,N)} &:= |u|_B^{(s)} + \sum_{s < |\beta| < r} \sup_{\mathbf{x} \in B \setminus N} \varrho_N^{|\beta| - s}(\mathbf{x}) |D^\beta u(\mathbf{x})| \\
&+ \sum_{|\beta| = [r]} \sup_{\mathbf{x} \in B \setminus N} \varrho_N^{r-s}(\mathbf{x}) \sup_{|\mathbf{x} - \mathbf{y}| < \frac{1}{2} \varrho_N(\mathbf{x})} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{r-[r]}}.
\end{aligned}$$

Clearly, $\dot{C}_s^r(B, N)$ is a subspace of $C_s^r(B, N)$ consisting of functions vanishing on N together with their derivatives of order up to $[s]$. For $s < 0$ assume $C_s^r(B, N) := \dot{C}_s^r(B, N)$.

Finally we define the weighted Hölder spaces to which the generalized solutions of Problem (II) belong. We use the following notations for some subdomains of Ω and \mathbb{R}_+^1 , respectively

$$\begin{aligned}
\Omega^0 &:= \{\mathbf{x} \in \Omega : |x_1| < 2\}, & \Omega^+ &:= \{\mathbf{x} \in \Omega : x_1 > 1\}, \\
\Omega^- &:= \{\mathbf{x} \in \Omega : x_1 < -1\}, & J^0 &:= (0, 2), & J^+ &:= (1, +\infty).
\end{aligned}$$

For an arbitrary real number $z > 0$ define the space

$$\begin{aligned}
C_{s,z}^r(\Omega) &= \{u(\mathbf{x}), u|_{\Omega^0} \in C_s^r(\Omega^0, Q), \exp(zx_1)u(\mathbf{x})|_{\Omega^+} \in C^r(\Omega^+), \\
&\exp(-zx_1)u(\mathbf{x})|_{\Omega^-} \in C^r(\Omega^-)\}
\end{aligned}$$

with the norm

$$\|u\|_{\Omega,s}^{r,z} := |u|_{C_s^r(\Omega^0, Q)} + |\exp(zx_1)u|_{\Omega^+}^{(r)} + |\exp(-zx_1)u|_{\Omega^-}^{(r)}.$$

Herein Q denotes the endpoint of Σ_3 (cf. Fig.2). For functions $f(x_1)$ defined in \mathbb{R}_+^1 we introduce the space $C_{s,z}^r(\mathbb{R}_+^1)$ with the norm

$$\|f\|_{\mathbb{R}_+^1,s}^{r,z} = |f|_{C_s^r(J^0, 0)} + |f(x_1) \exp(zx_1)|_{J^+}^{(r)}.$$

In the paper the spaces of scalar and vector-valued functions are not distinguished in notations. The norm for vector-valued functions is then the sum of the norms of the corresponding coordinate functions.

4. Solvability Results

Problem (I) can be handled by the same methods as in [13]. Let us start with the main result about this problem.

Theorem 1. *Let $S_2 = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = 1 + \varepsilon \varphi_2(x_1), -\infty < x_1 < +\infty\}$, $\varphi_2 \in W_\beta^{l+5/2, 2}(\mathbb{R})$ with $l \geq 0, \beta = \delta|\beta_0| > 0$, where β_0 is independent of δ and depends on eigenvalues of the operator pencils associated with the corresponding linear problem (cf. [12]). Assume that δ is sufficiently small. Then there exist positive numbers $\hat{\varepsilon}, \hat{r}$ such that for every $\varepsilon \in (0, \hat{\varepsilon})$ Problem (I) has a unique solution $(\mathbf{v}, p, \theta, \psi_1)^T$. The solution admits the representation*

$$\begin{aligned}
\mathbf{v}(\mathbf{x}) &= \mathbf{v}^0(\mathbf{x}) + \varepsilon \mathbf{u}(\mathbf{x}), & p(\mathbf{x}) &= p^0(\mathbf{x}) + \varepsilon q(\mathbf{x}), \\
\theta(\mathbf{x}) &= \theta^0(\mathbf{x}) + \varepsilon \vartheta(\mathbf{x}), & \psi_1(x_1) &= h_\infty + \varepsilon \Psi_1(x_1),
\end{aligned}$$

where $h_\infty \in (0, 1)$ is one of the roots to equation (5.4), $\{\mathbf{v}^0, p^0, \theta^0\}$ are the functions of the basic solution from (5.3), (5.5),

$$\mathbf{U} := (\mathbf{u}, q, \vartheta, \Psi_1)^T \in \left[W_\beta^{l+2,2}(II) \right]^2 \times W_\beta^{l+1,2}(II) \times W_\beta^{l+2,2}(II) \times W_\beta^{l+5/2,2}(\mathbb{R}) \\ \equiv \mathcal{D}_\beta^{l,2}W(II)$$

and the following inequalities hold

$$\|\mathbf{U}; \mathcal{D}_\beta^{l,2}W(II)\| \leq \hat{r}, \quad \hat{\varepsilon} \leq \text{const} \cdot \delta^2.$$

Let us remark that $\mathbf{U} = (\mathbf{u}, q, \vartheta, \Psi_1)^T$ in Theorem 1 is the unique solution of an associated linear boundary value problem that was obtained by linearization of the original Problem (I) over the basic solution $\{\mathbf{v}^0, p^0, \theta^0\}$ in the uniform unperturbed (strip-like) domain II . Note that the corresponding isothermal problem to Problem (I) (i.e. without any inclusion of temperature) was analytically examined in detail in the papers [12, 13]. In order to prove Theorem 1 one has to repeat and to modify all the investigations from that papers. Since the temperature equation is also nonlinear elliptic there are not essential changes in the proof. Thus we omit the detailed proof here.

Let us now study the solvability of Problem (II). By straightforward calculations one can determine the (exact) nonisothermal Poiseuille flows

$$\{\mathbf{v}^{(-)}(\mathbf{x}), p^{(-)}(\mathbf{x}), \theta^{(-)}(\mathbf{x})\} \quad \mathbf{x} \in \Omega^-$$

in the left part Ω^- of the (double) - channel. The corresponding velocities and temperatures do not depend on x_1 . In Ω_1^- (i.e. if $0 \leq x_2 \leq h_1$) one obtains

$$\left\{ \begin{aligned} v_1^{(-)}(x_2) &= \left(\frac{3R}{h_1^2} - \frac{6F_1}{h_1^3} \right) x_2^2 + \left(-\frac{4R}{h_1} + \frac{6F_1}{h_1^2} \right) x_2 + R, \\ v_2^{(-)}(\mathbf{x}) &\equiv 0, \quad \theta^{(-)}(x_2) = \theta_0 + x_2(\theta_3 - \theta_0), \\ p^{(-)}(\mathbf{x}) &= 2\nu_1 \varrho_1 \left(\frac{3R}{h_1^2} - \frac{6F_1}{h_1^3} \right) x_1 - \varrho_1 g x_2 + \\ &+ \varrho_1 \gamma_1 \left[\theta_0 x_2 + \frac{1}{2}(\theta_3 - \theta_0)x_2^2 \right] + k_1. \end{aligned} \right. \quad (4.1)$$

In Ω_2^- (i.e. if $h_1 \leq x_2 \leq 1$) one gets, respectively,

$$\left\{ \begin{aligned} v_1^{(-)}(x_2) &= -\frac{6F_2}{(1-h_1)^3} x_2^2 + \frac{6(1+h_1)F_2}{(1-h_1)^3} x_2 - \frac{6h_1F_2}{(1-h_1)^3}, \\ v_2^{(-)}(\mathbf{x}) &\equiv 0, \quad \theta^{(-)}(x_2) = \theta_3 + \frac{(x_2-h_1)}{(1-h_1)}(\theta_2 - \theta_3), \\ p^{(-)}(\mathbf{x}) &= -\frac{12\nu_2\varrho_2F_2}{(1-h_1)^3} x_1 - \varrho_2 g x_2 + \\ &+ \varrho_2 \gamma_2 \left[\theta_3 x_2 + \frac{1}{2}(\theta_2 - \theta_3) \frac{(x_2-h_1)^2}{1-h_1} \right] + k_2. \end{aligned} \right. \quad (4.2)$$

It is well-known that the pressure p can be determined only up to an additive constant in channel flows (cf. k_1, k_2 in formulae (4.1), (4.2)).

By $\{\mathbf{v}^{(+)}, p^{(+)}, \theta^{(+)}\}$ we denote the exact solution (nonisothermal Poiseuille flow) in the united part Ω^+ at the right-hand side of the channel. Note that this solution coincides with the basic solution $\{\mathbf{v}^0, p^0, \theta^0\}$ to Problem (I). That solution is determined by straightforward calculations in the Appendix (see Section 5). The associated flow fields are given by formulae (5.3), (5.5). Note that in [13] the corresponding isothermal flow fields were already calculated. An essential part of the determination of $\{\mathbf{v}^{(+)}, p^{(+)}, \theta^{(+)}\}$ consists in the calculation of the value h_∞ from the 5th degree polynomial equation (5.4). Equation (5.4) coincides with equation (A.13) from [13] when the channel is horizontal. Note that the final thickness h_∞ is a function of F_1, F_2, R and of the rheological parameters of the fluids. It can have up to three different values in the interval (0,1) for the same parameter set (cf. [13]). Furthermore, by $\psi_1^0(x_1)$ we denote the infinitely differentiable solution of the following boundary value problem

$$\begin{cases} \frac{d}{dx_1} \frac{\psi_1'(x_1)}{\sqrt{1 + \psi_1'(x_1)^2}} - \frac{g(\varrho_1 - \varrho_2)}{\sigma(0)} \psi_1(x_1) = -\frac{g(\varrho_1 - \varrho_2)}{\sigma(0)} h_\infty, \\ \psi_1(0) = h_1, \quad \lim_{x_1 \rightarrow +\infty} \psi_1(x_1) = h_\infty, \end{cases} \quad (4.3)$$

which can be obtained from the sixth condition (1.6)₆ of (1.6) by setting $\mathbf{v} = \mathbf{0}$, $p = \text{const.}$, $\theta = 0$ as the initial solution for $F_1 = F_2 = R = \theta_0 = \theta_2 = 0$. Let $\xi = \xi(x_1)$ be a smooth cut-off function vanishing for $|x_1| \leq 1$ and being equal to 1 for $|x_1| \geq 2$. Finally, suppose that $\varrho_1 > \varrho_2$ is fulfilled. Now we can formulate the main result for Problem (II).

Theorem 2. *There exist positive real numbers*

$$s_0, M_0 \quad \text{and} \quad z_0 \leq \sqrt{g(\varrho_1 - \varrho_2)/\sigma(0)}$$

such that for arbitrary $s \in (0, s_0)$, $z \in (0, z_0)$, $\max(F_1, F_2, R, |\theta_0|, |\theta_2|) < M_0$ and for positive h_∞, F_1, F_2, R satisfying the condition

$$|h_1 - h_\infty(F_1, F_2, R)| < \sqrt{\frac{2\sigma(0)}{g(\varrho_1 - \varrho_2)}}, \quad (4.4)$$

Problem (II) has a unique solution $\{v, p, \theta, \psi_1\}$ which can be represented in the form

$$\begin{aligned} \mathbf{v} &= \xi(-x_1)\mathbf{v}^{(-)} + \xi(x_1)\mathbf{v}^0 + \mathbf{w}, & p &= \xi(-x_1)p^{(-)} + p^0 + q, \\ \theta &= \xi(-x_1)\theta^{(-)} + \xi(x_1)\theta^0 + \vartheta, & \psi_1(x_1) &= \psi_1^0(x_1) + \omega(x_1), \end{aligned}$$

where $\{\mathbf{v}^{(-)}, p^{(-)}, \theta^{(-)}\}$ denotes the nonisothermal Poiseuille flow in both channels as $x_1 \rightarrow -\infty$ and $\{\mathbf{v}^0, p^0, \theta^0\}$ is the basic solution of Problem (I)

as $x_1 \rightarrow +\infty$. The function ξ denotes the smooth cut-off function mentioned above. Moreover, $\mathbf{w} \in [C_{s,z}^{s+2}(\Omega)]^2$, $\vartheta \in C_{s,z}^{s+2}(\Omega)$, $q \in C_{s-1,z}^{s+1}(\Omega^0 \cup \Omega^+)$, $\nabla q \in C_{s-2,z}^s(\Omega)$ and $\omega \in C_{1+s,z}^{3+s}(\mathbb{R}_+^1)$ hold.

The proof of this theorem can be realized in the same way as in [17] applying the above mentioned scheme (2.2). We omit here the proof. The condition (4.4) is a consequence of solving the boundary value problem (4.3) and the restriction $\varrho_1 > \varrho_2$ is essential for the applied method. The weight parameter s_0 in Theorem 2 can be estimated studying a model problem for a nonisothermal Stokes system in a neighborhood of Q in the same way as in [17, 18] and [21]. The exponential decay of $\{\mathbf{w}, q, \vartheta, \omega\}$ at infinity is well-known (cf. [9, 17, 18]).

5. Appendix - On the Basic Uniform Solutions

Let us consider now the two-fluid flow in the “uniform” horizontal channel Π , i.e. in the case where S_2 is the straight line $S_2 := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 \equiv 1, -\infty < x_1 < +\infty\}$. We suppose that the (unknown) free interface Γ_1 permits the representation $\Gamma_1 = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \psi_1(x_1) = h_\infty = \text{const.}, -\infty < x_1 < +\infty\}$ and we are looking for stationary unidirectional flows. Such flows fulfill the assumptions

$$v_2 \equiv 0, \quad \frac{\partial v_1}{\partial x_1} \equiv 0, \quad \frac{\partial \theta}{\partial x_1} \equiv 0.$$

As a consequence we obtain that the pressure gradient downstream is an unknown constant $\partial p / \partial x_1 = p_0 = \text{const.}$ which is determined by prescribing volume fluxes F_1, F_2 . Under these assumptions the nonisothermal Navier-Stokes equations reduce to

$$\begin{aligned} -\nu_\varrho \nabla^2 v_1 + p_0 &= 0, \\ (\partial p / \partial x_2) &= -\varrho g + \varrho \gamma \theta, \quad -\lambda \nabla^2 \theta = 0, \end{aligned}$$

and the equation of continuity (1.1)₂ is automatically fulfilled. Problem (I) can now be transformed to the following two independent systems of equations containing the unknowns $v_1(x_2), p_0, h_\infty$ and $\theta(x_2), p(x_1, x_2)$, respectively

$$\begin{aligned} \nu_1 \varrho_1 \frac{d^2 v_1^{(1)}}{dx_2^2} &= p_0, & \nu_2 \varrho_2 \frac{d^2 v_1^{(2)}}{dx_2^2} &= p_0, \\ v_1^{(1)}(0) &= R, & v_1^{(2)}(1) &= 0, \\ v_1^{(1)}|_{x_2=h_\infty} &= v_1^{(2)}|_{x_2=h_\infty}, & \nu_1 \varrho_1 \left. \frac{dv_1^{(1)}}{dx_2} \right|_{x_2=h_\infty} &= \nu_2 \varrho_2 \left. \frac{dv_1^{(2)}}{dx_2} \right|_{x_2=h_\infty}, \quad (5.1) \\ \int_0^{h_\infty} v_1^{(1)}(x_2) dx_2 &= F_1, & \int_{h_\infty}^1 v_1^{(2)}(x_2) dx_2 &= F_2, \end{aligned}$$

$$\begin{aligned}
\frac{d^2\theta^{(1)}}{dx_2^2} &= 0, & \frac{d^2\theta^{(2)}}{dx_2^2} &= 0, \\
\theta^{(1)}|_{x_2=0} &= \theta_0, & \theta^{(2)}|_{x_2=1} &= \theta_2, \\
\theta^{(1)}|_{x_2=h_\infty} &= \theta^{(2)}|_{x_2=h_\infty}, & \lambda_1 \frac{d\theta^{(1)}}{dx_2}\Big|_{x_2=h_\infty} &= \lambda_2 \frac{d\theta^{(2)}}{dx_2}\Big|_{x_2=h_\infty}, \\
\frac{\partial p}{\partial x_2} &= -\varrho g + \varrho\gamma\theta(x_2), & p^{(1)}|_{x_2=h_\infty} &= p^{(2)}|_{x_2=h_\infty},
\end{aligned} \tag{5.2}$$

In the systems (5.1) and (5.2) the notation $v_1^{(m)}$ means the restriction $v_1|_{\Pi_m}$ of the field v_1 to the subdomain Π_m ($m = 1, 2$). The analogous statements are true for $v_2^{(m)}$, $\theta^{(m)}$, $p^{(m)}$, respectively. System (5.1) is the same as in [13] and it has the well-known solution

$$\begin{aligned}
v_1^0(x_2) &= \begin{cases} 0.5a_1x_2^2 + b_1x_2 + R, & 0 \leq x_2 \leq h_\infty \\ 0.5a_2(x_2^2 - 1) + b_2(x_2 - 1), & h_\infty \leq x_2 \leq 1 \end{cases}, \\
v_2^0(x_2) &\equiv 0,
\end{aligned} \tag{5.3}$$

$$p_0 = a_1\nu_1\varrho_1 = a_2\nu_2\varrho_2,$$

where

$$\begin{aligned}
r &= \frac{\nu_1\varrho_1}{\nu_2\varrho_2}, & a_1 &= \left[-3\frac{F_1 - Rh_\infty}{h_\infty^2} - 3\frac{F_2}{r(1 - h_\infty^2)} \right], & a_2 &= r a_1, \\
b_1 &= \left[(2 + h_\infty)\frac{F_1 - Rh_\infty}{h_\infty^2} + h_\infty\frac{F_2}{r(1 - h_\infty^2)} \right], & b_2 &= r b_1,
\end{aligned}$$

and h_∞ is given by one of the solutions to the following equation (5.4) which is an algebraic equation of the fifth degree (see [13])

$$\begin{aligned}
&r(r-1)Rh_\infty^5 + [-4r(r-1)R - r(r-1)F_1 - (r-1)F_2]h_\infty^4 \\
&+ [r(6r-5)R + 2r(2r-3)F_1 - 2rF_2]h_\infty^3 + [2r(-2r+1)R \\
&+ 3r(-2r+3)F_1 + 3rF_2]h_\infty^2 + [r^2R + 4r(r-1)F_1]h_\infty - r^2F_1 = 0.
\end{aligned} \tag{5.4}$$

In [13] the following two lemmas on the existence and multiplicity of solutions to equation (5.4) were proved.

Lemma 1. *If $F_1F_2 > 0$, then equation (5.4) has at least one root h_∞ within the open interval $(0, 1)$.*

Lemma 2. *If $F_1F_2 \geq 0$, then equation (5.4) has at most three different roots $h_\infty \in (0, 1)$.*

If h_∞ and p_0 are known from system (5.1) then system (5.2) is independent of (5.1) and can be solved separately. Its solution has the representation

$$\begin{aligned} \theta^0(x_2) &= \begin{cases} \theta_0 + \frac{x_2}{h_\infty} (\theta_\infty - \theta_0), & 0 \leq x_2 \leq h_\infty \\ \theta_\infty + \frac{x_2 - h_\infty}{1 - h_\infty} (\theta_2 - \theta_\infty), & h_\infty \leq x_2 \leq 1 \end{cases}, \\ p^0(\mathbf{x}) &= \begin{cases} p_0 x_1 - \varrho_1 g x_2 + \varrho_1 \gamma_1 \left(\theta_0 x_2 + \frac{\theta_\infty - \theta_0}{2h_\infty} x_2^2 \right) + k, & \\ p_0 x_1 - \varrho_2 g x_2 + \varrho_2 \gamma_2 \left(\frac{\theta_\infty - h_\infty \theta_2}{1 - h_\infty} x_2 + \frac{\theta_2 - \theta_\infty}{2(1 - h_\infty)} x_2^2 \right) \\ + p_{\text{corr}} + k, \end{cases} \end{aligned} \quad (5.5)$$

Note that in Problem (I) the pressure p can be determined only up to an additive constant k . This fact is well-known for channel flows.

Furthermore, in (5.5) the correction term p_{corr} results from the last condition (5.2)₈ in (5.2) and θ_∞ denotes the a priori unknown value of θ at $x_2 = h_\infty$. These values are given by

$$\begin{aligned} p_{\text{corr}} &= gh_\infty(\varrho_2 - \varrho_1) + \varrho_1 \gamma_1 h_\infty \frac{\theta_0 + \theta_\infty}{2} - \varrho_2 \gamma_2 \frac{h_\infty}{2(1 - h_\infty)} (2\theta_\infty - \theta_2 h_\infty - \theta_\infty h_\infty), \\ \theta_\infty &= \frac{\lambda_2 \theta_2 h_\infty + \lambda_1 \theta_0 (1 - h_\infty)}{\lambda_1 (1 - h_\infty) + \lambda_2 h_\infty}. \end{aligned}$$

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