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MULTIPLE SOLUTIONS OF THE BVB FOR TWO-DIMENSIONAL SYSTEM BY EXTRACTING LINEAR PARTS AND QUASILINEARIZATION[1]

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Abstract. We investigate a two-dimensional differential system of the form $x' = f(t, y)$, $y' = h(t, x)$ together with the boundary conditions $x(0) = 0$, $x(1) = 0$ by using the quasilinearization process. We show that if this problem allows for quasilinearization with respect to essentially different linear parts, then it has multiple solutions.

Key words: quasilinearization, quasi-linear system, i-nonresonant linear part, i-type solution, Green's matrix.

1 Introduction

In this paper we investigate the multiplicity of solutions of nonlinear boundary value problem (BVP) for two-dimensional differential system of the form

$$\begin{cases} x' = f(t, y), \\ y' = h(t, x), \qquad\qquad t \in I := [0, 1], \quad f, h \in C\big(I \times \mathbb{R}; \mathbb{R}\big). \\ x(0) = 0, \quad x(1) = 0, \end{cases} \tag{1.1}$$

Our research is motivated by the papers of R. Conti [1], L. Jackson and K. Schrader [2], who studied (among other things) oscillatory properties of solutions of two-point boundary value problems.

We investigate solvability of the BVP (1.1) applying for quasilinearization process described in [3, 4, 5]. We try to transform a nonlinear system in (1.1)

[1] This work was advised by prof. F. Sadyrbaev (Institute of Mathematics and Computer Science, University of Latvia).

to a certain quasi-linear one. First, a linear part is extracted, for instance:

$$\begin{cases} x' - k\,y = f(t,\,y) - k\,y, \\ y' + k\,x = h(t,\,x) + k\,x. \end{cases} \tag{1.2}$$

Suppose that a coefficient k satisfies $\sin k \neq 0$, then the extracted linear part $\big(LX\big)(t) := \begin{pmatrix} x' - k\,y \\ y' + k\,x \end{pmatrix}$ is non-resonant with respect to the given boundary conditions

$$x(0) = 0, \quad x(1) = 0 \tag{1.3}$$

or, equivalently, the respective homogeneous problem

$$\begin{cases} x' - k\,y = 0, \\ y' + k\,x = 0, \end{cases} \qquad x(0) = 0, \quad x(1) = 0 \tag{1.4}$$

has only the trivial solution.

Then we wish to make bounded the right sides in the system (1.2), for instance, we consider truncated functions $F_k(t, y) = \overline{f(t,\,y) - k\,y}$ and $H_k(t, y) = \overline{h(t,\,x) + k\,x}$, which coincide with the earlier obtained functions $f(t,\,y) - k\,y$ and $h(t,\,x) + k\,x$ respectively in some domain $\Omega_k = \{(t,\,x,\,y) : \ 0 \leq t \leq 1, \ |x| \leq N_x, \ |y| \leq N_y\}$. Thus the original nonlinear system in (1.1) and modified quasi-linear one

$$\begin{cases} x' - k\,y = F_k(t,\,y), \\ y' + k\,x = H_k(t,\,x), \end{cases} \tag{1.5}$$

are equivalent in a domain Ω_k. If any solution of the quasi-linear problem (1.5), (1.3) satisfies the inequalities $|x(t)| \leq N_x$, $|y(t)| \leq N_y \ \ \forall t \in [0,\,1]$, then we say that the original problem (1.1) allows for quasilinearization with respect to the extracted linear part $\big(LX\big)(t)$.

If a solution $\big(x(t),\,y(t)\big)$ of the quasi-linear problem (1.5), (1.3) is located in the domain of equivalence Ω_k, then this solution also solves the original nonlinear problem (1.1). Notice that an oscillatory type of a solution $\big(x(t),\,y(t)\big)$ corresponds to a type of non-resonance of the extracted linear part $\big(LX\big)(t)$.

If the original problem allows for quasilinearization with respect to another *essentially different* linear part (i.e. with different type of non-resonance), then the problem (1.1) is expected to have multiple solutions.

2 Quasi-Linear Systems and Types of Solutions

Consider the quasi-linear system

$$\begin{cases} x' - ky &= F_1(t, y), \\ y' + kx &= F_2(t, x), \end{cases} \tag{2.1}$$

where functions F_1, F_2 are continuous, bounded and satisfy the Lipschitz conditions with respect to y and x respectively, together with the given boundary conditions (1.3). Suppose that the extracted linear part

$$\big(LX\big)(t) := \begin{pmatrix} x' - k\,y \\ y' + k\,x \end{pmatrix}$$

is non-resonant with respect to the boundary conditions under consideration. In order to classify the linear parts with respect to the boundary conditions (1.3) for different values of k consider the homogeneous problem (1.4).

Let us introduce polar coordinates as $x(t)=r(t)\sin\varphi(t)$, $y(t)=r(t)\cos\varphi(t)$. Then the angular function $\varphi(t)$ for homogeneous system in (1.4) satisfies $\varphi'(t) = k$ and therefore $\varphi(t)$ is monotonically increasing if $k > 0$.

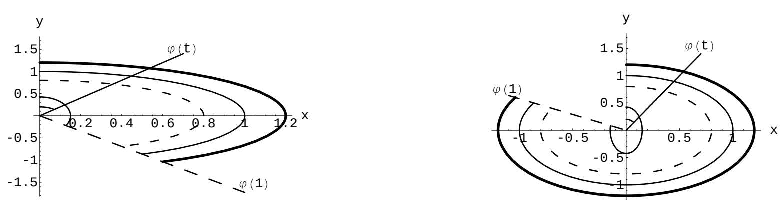
The boundary conditions (1.3) in polar coordinates take the form

$$\varphi(0) = 0, \quad \varphi(1) = \pi\, n, \quad n \in \mathbb{N}. \tag{2.2}$$

A linear part $\big(LX\big)(t)$ under consideration is non-resonant with respect to the boundary conditions (1.3), if coefficient k belongs to one of the following intervals $(0,\, \pi)$, $(\pi,\, 2\pi)$, $(2\pi,\, 3\pi), \ldots, (i\pi,\, (i+1)\pi), \ldots,\ i \in \mathbb{N} \cup \{0\}$. In each of these intervals of non-resonance the angular function $\varphi(t)$ has distinctive properties. We can illustrate this fact considering the phase portraits of solutions to a Cauchy problem

$$\begin{cases} x' - k\,y = 0, \\ y' + k\,x = 0, \end{cases} \qquad x(0) = 0, \quad y(0) = A \tag{2.3}$$

in the interval $t \in [0,\, 1]$ for different values of k (see Fig. 1).



a) $k = \frac{5\pi}{6}$ b) $k = \frac{5\pi}{3}$

Figure 1. Phase portraits of solutions to problem (2.3) in the interval $t \in [0,\, 1]$.

If $k \in (0,\, \pi)$ then the angular function $\varphi(t)$ of solutions to the problem (2.3) in the interval $t \in (0,\, 1]$ does not take values of the form πn, $n \in \mathbb{N}$ (see Fig. 1a). If $k \in (\pi,\, 2\pi)$ then the angular function $\varphi(t)$ in the interval $t \in (0,\, 1]$ takes exactly one value of the form πn, $n \in \mathbb{N}$ (see Fig. 1b).

Definition 1. The linear part $\big(LX\big)(t)$ is called an *i-nonresonant* with respect to the boundary conditions (1.3), if the angular function $\varphi(t)$ of solutions of the system $\big(L\,X\big)(t) = 0$, defined by the initial condition $\varphi(0) = 0$, takes exactly i values of the form πn in the interval $(0,\, 1)$ and $\varphi(1) \neq \pi n$, $n \in \mathbb{N}$.

Proposition 1. *The linear part* $\big(LX\big)(t)$ *is i-nonresonant with respect to the boundary conditions* (1.3)*, if* $k \in (i\pi,\, (i+1)\pi)$, $i \in \mathbb{N} \cup \{0\}$.

If the numbers k_i and k_j belong to different intervals of non-resonance then the respective linear parts have different types of non-resonance, in this case we say for brevity that the linear parts are *essentially different*.

Since the linear part in (2.1) is non-resonant with respect to the boundary conditions (1.3) and functions F_1, F_2 are bounded, then in accordance to the Conti theorem [1] the BVP (2.1), (1.3) is solvable. Let $\big(\xi(t), \eta(t)\big)$ be a solution of the quasi-linear problem (2.1), (1.3).

Definition 2. We say that $\big(x(t;\delta),\, y(t;\delta)\big)$ is a neighbouring solution of a solution $\big(\xi(t), \eta(t)\big)$, if $\big(x(t;\delta),\, y(t;\delta)\big)$ solves the same quasi-linear system (2.1), satisfies the first boundary condition $x(0;\delta) = 0$ and there exists $\varepsilon > 0$ such that $\forall\, \delta \in (0, \varepsilon]$ $y(0;\delta) = \eta(0) + \delta \operatorname{sgn} \eta(0)$.

In order to classify solutions of the quasi-linear problem (2.1), (1.3) let us introduce (local) polar coordinates for the difference between neighbouring solution $\big(x(t;\delta), y(t;\delta)\big)$ and investigated solution $\big(\xi(t), \eta(t)\big)$ as

$$\begin{aligned} x(t;\delta) - \xi(t) &= \rho(t)\, \sin \Theta(t;\delta), \\ y(t;\delta) - \eta(t) &= \rho(t)\, \cos \Theta(t;\delta), \end{aligned} \qquad \Theta(0;\delta) = 0. \tag{2.4}$$

Definition 3. We say that $\big(\xi(t), \eta(t)\big)$ is an i-type solution of the problem (2.1), (1.3) if there exists $\varepsilon > 0$ such that $\forall \delta \in (0, \varepsilon]$ the angular function $\Theta(t;\delta)$, defined by formulas (2.4), takes exactly i values of the form πn in the interval (0, 1) and $\Theta(1;\delta) \neq \pi n$, $n \in \mathbb{N}$.

3 Main Result for Quasi-Linear Problems

Consider the quasi-linear problem (2.1), (1.3), where the extracted linear part is non-resonant with respect to the boundary conditions (1.3) and functions F_1, F_2 are continuous, bounded and satisfy the Lipschitz conditions with respect to y and x respectively. The last assumption ensures the unique solvability of the Cauchy problems (2.1), $x(0) = A,\ y(0) = B$ and, as a consequence, continuous dependence of solutions on the initial data.

Lemma 1. *A set of solutions to the quasi-linear problem* (2.1)*,* (1.3) *is non-empty and compact in* $C_2^1([0, 1], R)$.

The proof is standard, using the Green's matrix approach and the Arzela - Ascoli criterium.

Corollary 1. A set of initial values $(0, y(0))$ of solutions to the problem (2.1), (1.3) is compact in R.

Theorem 1. *If the extracted linear part* $\big(LX\big)(t)$ *in the quasi-linear system* (2.1) *is i-nonresonant with respect to the boundary conditions* (1.3)*, then the problem* (2.1)*,* (1.3) *has an i-type solution.*

This theorem was proved in [5, 6] for a quasi-linear system with a linear part of more complicated form.

4 BVP for Nonlinear System

Consider a two-dimensional nonlinear system

$$\begin{cases} x' = r(t)|y|^p \operatorname{sgn} y, \\ y' = -q(t)|x|^p \operatorname{sgn} x, \end{cases} \tag{4.1}$$

where $t \in I := [0,\, 1]$, $p > 1$, $r, q \in C\big(I;\, (0,\, +\infty)\big)$ with the given boundary conditions (1.3). Suppose that $0 < r_1 \le r(t) \le r_2$, $0 < q_1 \le q(t) \le q_2$, $\forall t \in [0,\, 1]$. This system is equivalent to a system

$$\begin{cases} x' - k\,y = r(t)|y|^p \operatorname{sgn} y - k\,y =: u(t,\, y), \\ y' + k\,x = k\,x - q(t)|x|^p \operatorname{sgn} x =: v(t,\, x), \end{cases} \tag{4.2}$$

where the coefficient $k > 0$ satisfies $\sin k \ne 0$. Let

$$\delta(x, y, z) = \begin{cases} x, & y < x, \\ y, & x \le y \le z, \\ z, & y > z. \end{cases}$$

Consider quasi-linear system

$$\begin{cases} x' - ky = U_k(t,\, y) := u\big(t,\, \delta(-N_y, y, N_y)\big), \\ y' + kx = V_k(t,\, x) := v\big(t, \delta(-N_x, x, N_x)\big). \end{cases} \tag{4.3}$$

Due to properties of functions $u(t,\, y)$ and $v(t,\, y)$ the right sides in (4.3) are bounded and satisfy the Lipschitz conditions, thus Theorem 1 is applicable.

The following assertions are evident.

Proposition 2. *The problem* (4.3), (1.3) *is solvable. Moreover, it has an i-type solution if* $k \in (i\pi,\, (i+1)\pi)$, $i \in \mathbb{N} \cup \{0\}$.

Proposition 3. *If a solution* $\big(x(t), y(t)\big)$ *of the problem* (4.3), (1.3) *satisfies the inequalities*

$$|x(t)| < N_x, \quad |y(t)| < N_y, \quad \forall t \in [0, 1], \tag{4.4}$$

then it solves also the original problem (4.1), (1.3).

If any solution of the problem (4.3), (1.3) satisfies the estimates (4.4) then we say that nonlinear problem (4.1), (1.3) allows for quasilinearization (with respect to the extracted linear part). Any solution $\big(x_k(t), y_k(t)\big)$ of the problem (4.3), (1.3) can be written in the integral form

$$\begin{cases} x_k(t) = \int\limits_0^1 \Big(G_k^{11}(t,s)\, U_k\big(s, y(s)\big) + G_k^{12}(t,s)\, V_k\big(s, x(s)\big)\Big)\, ds, \\ y_k(t) = \int\limits_0^1 \Big(G_k^{21}(t,s)\, U_k\big(s, y(s)\big) + G_k^{22}(t,s)\, V_k\big(s, x(s)\big)\Big)\, ds, \end{cases} \tag{4.5}$$

where $G_k^{ij}(t,\, s)$ $(i,j = 1,2)$ are the elements of the Green's matrix of the respective homogeneous problem (1.4). Then we get the following estimates

$$\begin{cases} |x_k(t)| \le \Gamma_{11}(k) M_y + \Gamma_{12}(k) M_x, \\ |y_k(t)| \le \Gamma_{21}(k) M_y + \Gamma_{22}(k) M_x, \end{cases} \tag{4.6}$$

where $\Gamma_{ij}(k)$ $(i,j = 1,2)$ are the estimates of the respective elements $G_k^{ij}(t,\, s)$ of the Green's matrix and $M_y = \sup |U_k(t,y)|$, $M_x = \sup |V_k(t,x)|$.

Proposition 4. *If the inequalities*

$$\begin{cases} \Gamma_{11}(k) M_y + \Gamma_{12}(k) M_x \;<\; N_x, \\ \Gamma_{21}(k) M_y + \Gamma_{22}(k) M_x \;<\; N_y \end{cases} \tag{4.7}$$

hold then the nonlinear problem (4.1), (1.3) *allows for quasilinearization and therefore it has a solution of definite type.*

Theorem 2. *If there exists some number* $k_i \in \big(i\pi, (i+1)\pi\big)$, $i \in \mathbb{N} \cup \{0\}$, *which satisfies the inequality*

$$\frac{k_i}{|\sin k_i|}\, p^{\frac{p}{1-p}}\, (p-1) \Big(r_1^{\frac{1}{1-p}} + q_1^{\frac{1}{1-p}} \Big) < \gamma\, A, \tag{4.8}$$

where γ *is a root of the equation* $\gamma^p = \gamma + (p-1) p^{\frac{p}{1-p}}$, $A = \min\{ r_2^{\frac{1}{1-p}}, q_2^{\frac{1}{1-p}} \}$, *then there exists an* i*-type solution of the nonlinear problem* (4.1), (1.3).

Proof. The given nonlinear system (4.1) is equivalent to quasi-linear one (4.3) in a domain $\Omega_k = \{(t,\, x,\, y) :\ \ 0 \le t \le 1,\ |x| \le N_x,\ |y| \le N_y\}$. Positive numbers N_x, N_y are chosen as the x and y values, where the functions $v(t,\, x)$ and $u(t,\, y)$ respectively take the values opposite to their local extremum. (In detail such approach was considered in [3].) Computation gives that

$$N_x = \left(\frac{k}{q_2} \right)^{\frac{1}{p-1}} \gamma, \quad N_y = \left(\frac{k}{r_2} \right)^{\frac{1}{p-1}} \gamma, \tag{4.9}$$

where γ is a root of the equation $\gamma^p = \gamma + (p-1)\, p^{\frac{p}{1-p}}$, and

$$\begin{aligned} M_x &= \sup_{\Omega_k} |V_k(t,x)| = \left(\frac{k}{p} \right)^{\frac{p}{p-1}} q_1^{\frac{1}{1-p}}\, (p-1), \\ M_y &= \sup_{\Omega_k} |U_k(t,y)| = \left(\frac{k}{p} \right)^{\frac{p}{p-1}} r_1^{\frac{1}{1-p}}\, (p-1). \end{aligned} \tag{4.10}$$

Since the Green's matrix of the respective homogeneous linear problem (1.4) is given by

$$\mathbb{G}_k(t,\,s)=\begin{cases}\dfrac{1}{\sin k}\begin{pmatrix}-\cos(ks)\sin(k(t-1)) & \sin(ks)\sin(k(t-1))\\ -\cos(ks)\cos(k(t-1)) & \sin(ks)\cos(k(t-1))\end{pmatrix} \\ \qquad\qquad\qquad\qquad \text{if} \quad 0\le s\le t\le 1,\\ \dfrac{1}{\sin k}\begin{pmatrix}-\sin(kt)\cos(k(s-1)) & \sin(kt)\sin(k(s-1))\\ -\cos(kt)\cos(k(s-1)) & \cos(kt)\sin(k(s-1))\end{pmatrix} \\ \qquad\qquad\qquad\qquad \text{if} \quad 0\le t< s\le 1,\end{cases} \tag{4.11}$$

therefore the elements $G_k^{ij}(t,\,s)$ of Green's matrix satisfy the same estimate

$$\left|G_k^{ij}(t,\,s)\right|\le\frac{1}{|\sin k|}=:\mathit{\Gamma}_k,\quad (i,j=1,2). \tag{4.12}$$

It follows from (4.9), (4.10), (4.12) that inequalities (4.7) reduce to (4.8).

If inequality (4.8) holds then the original problem (4.1), (1.3) allows for quasilinearization with respect to the linear part given above and, as a consequence, it has a solution of definite type. If moreover the inequality (4.8) is fulfilled for $k_i\in\big(i\pi,(i+1)\pi\big)$, $i\in\mathbb{N}\cup\{0\}$ then a solution of the problem (4.1), (1.3) will be an i-type solution, because the linear part in this case is i-nonresonant with respect to the boundary conditions (1.3). □

Corollary 2. If there exist numbers $k_i\in\big(i\pi,(i+1)\pi\big)$, $i=0,1,2,\dots,m$, which satisfy the inequality (4.8), then there exist at least $m+1$ solutions of different types to the BVP (4.1), (1.3).

Notice that if the problem (4.1), (1.3) has a non-trivial solution $\big(x(t),\,y(t)\big)$ then $\big(-x(t),\,-y(t)\big)$ also is a solution of this problem.

Corollary 3. If there exist numbers $k_i\in\big(i\pi,(i+1)\pi\big)$, $i=0,1,2,\dots,m$, which satisfy the inequality (4.8), then there exist at least $2m+1$ different solutions of the BVP (4.1), (1.3).

Let us denote

$$\begin{aligned}\mu&=\frac{r_1}{q_2},\quad \text{if}\ \begin{cases}r_1<q_1\\ r_2<q_2\end{cases},\qquad \text{or}\quad \mu=\frac{q_1}{r_2},\quad\text{if}\ \begin{cases}r_1>q_1\\ r_2>q_2\end{cases},\\ \mu&=\frac{r_1}{r_2},\quad \text{if}\ \begin{cases}r_1\le q_1\\ r_2\ge q_2\end{cases},\qquad \text{or}\quad \mu=\frac{q_1}{q_2},\quad\text{if}\ \begin{cases}r_1\ge q_1\\ r_2\le q_2\end{cases}.\end{aligned} \tag{4.13}$$

Then we notice that inequality (4.8) is fulfilled if the following inequality holds:

$$\frac{2k_i}{|\sin k_i|}\,p^{\frac{p}{1-p}}\,(p-1)\,\mu^{\frac{1}{1-p}}<\gamma. \tag{4.14}$$

Notice that the expression $\dfrac{k}{|\sin k|}$ takes the minimum value at the point, which is a root of an equation $k = \tan k$. So k_i can be chosen as that root of the equation above, which belongs to the interval $\left(i\pi, \dfrac{(2i+1)\pi}{2}\right)$, $i \in \mathbb{N} \cup \{0\}$. The results of calculations are provided in the Table 1. For certain values of p and μ this table shows that certain k_i in the form mentioned above satisfy the inequality (4.14). This table may be interpreted as a set of multiplicity results for the BVP (4.1), (1.3), since a lower index i of the coefficient k_i indicates that the BVP under consideration has an i-type solution.

Table 1. Results of calculations.

p	γ	μ	k_i
$\frac{3}{2}$	1.2509	$\mu \geq 0.8390$	$k_0;\ k_1$
$\frac{4}{3}$	1.2703	$\mu \geq 0.9144$	$k_0;\ k_1$
$\frac{5}{4}$	1.2813	$\mu \geq 0.8760$	$k_0;\ k_1$
		$\mu \geq 0.9991$	$k_0;\ k_1;\ k_2$
$\frac{6}{5}$	1.2884	$\mu \geq 0.8630$	$k_0;\ k_1$
		$\mu \geq 0.9588$	$k_0;\ k_1;\ k_2$
$\frac{7}{6}$	1.2933	$\mu \geq 0.8596$	$k_0;\ k_1$
		$\mu \geq 0.9384$	$k_0;\ k_1;\ k_2$
		$\mu \geq 0.9931$	$k_0;\ k_1;\ k_2;\ k_3$

5 Example

Consider the problem

$$\begin{cases} x' = 0.1\left(24 + 0.6\sin\dfrac{\pi}{2}t\right)|y|^{\frac{6}{5}}\operatorname{sgn} y, \\ y' = -\left(2.4 + \dfrac{0.2}{\pi}\operatorname{arc\,tan}(1-t)\right)|x|^{\frac{6}{5}}\operatorname{sgn} x, \end{cases} \qquad x(0) = x(1) = 0, \quad (5.1)$$

that is a special case of problem (4.1), (1.3) with

$$p = \frac{6}{5}, \quad r(t) = 0.1\left(24 + 0.6\sin\frac{\pi}{2}t\right), \quad q(t) = \left(2.4 + \frac{0.2}{\pi}\operatorname{arc\,tan}(1-t)\right).$$

For any $t \in [0, 1]$ we have that $2.4 \leq r(t) \leq 2.46$ and $2.4 \leq q(t) \leq 2.45$, therefore it is a case $r_1 = q_1$ and $r_2 > q_2$, thus $\mu = \dfrac{r_1}{r_2}$, $\mu = 0.97561$. In accordance with calculations (see Table 1) the BVP (5.1) allows for at least three essentially different quasilinearizations and therefore there exist at least five different solutions of this problem. We have computed all of them.

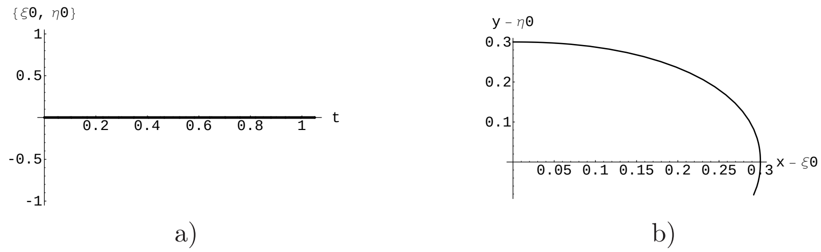


Figure 2. 0-type solution $(\xi_0(t), \eta_0(t))$ of the problem (5.1): a) the trivial solution $\xi_0(t) \equiv 0,\ \eta_0(t) \equiv 0$; b) phase portrait of the difference between neighboring solution $(x(t;\delta),\ y(t;\delta))$ and $(\xi_0(t),\ \eta_0(t)),\ \ t \in [0,\ 1]$, if $\delta = 0.3$.

Fig. 2*a* illustrates the trivial solution $(\xi_0(t),\ \eta_0(t))$ of problem (5.1), which is a 0-type solution, because an angular function of the difference between neighboring solution and trivial one in the interval $t \in (0,\ 1]$ does not take value of the form $\pi\, n,\ n \in \mathbb{N}$ (see Fig. 2*b*).

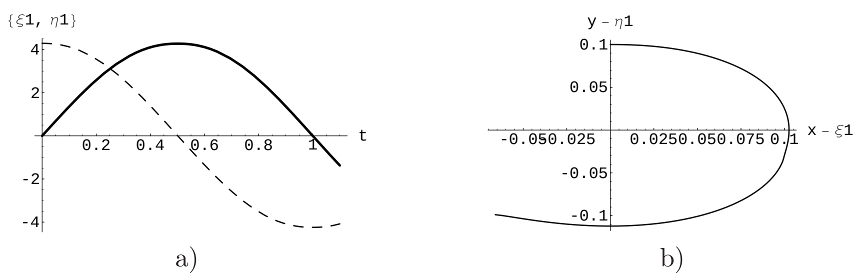
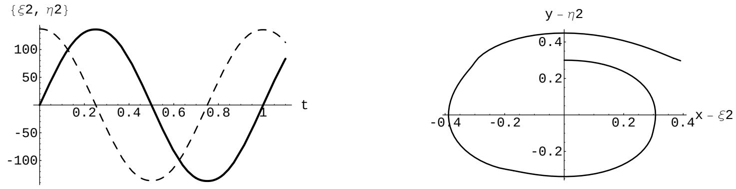


Figure 3. 1-type solution $(\xi_1(t), \eta_1(t))$ of the problem (5.1): a) initial data $\xi_1(0) = 0,\ \eta_1(0) = 4.2886$; b) phase portrait of the difference between neighboring solution $(x(t;\delta),\ y(t;\delta))$ and $(\xi_1(t),\ \eta_1(t)),\ \ t \in [0,\ 1]$, if $\delta = 0.1$.

Both lines (solid and dashed) in Fig. 3*a* indicate the second solution $(\xi_1(t),\ \eta_1(t))$ of the problem (5.1). Since an angular function of the difference between neighboring solution and this solution in the interval $t \in (0,\ 1]$ takes exactly once a value of the form $\pi\, n,\ n \in \mathbb{N}$ (see Fig. 3*b*), then $(\xi_1(t),\ \eta_1(t))$ is an 1-type solution.

Fig. 4*a* illustrates another solution of the problem (5.1). A solution $(\xi_2(t),\ \eta_2(t))$ is a 2-type solution, because an angular function of the difference between neighboring solution and this one in the interval $t \in (0,\ 1]$ takes values of the form $\pi\, n,\ n \in \mathbb{N}$ exactly two times (see Fig. 4*b*).

Fig. 5 and Fig. 6 show another two solutions of the problem (5.1), which are 1-type and 2-type solutions respectively.



a) b)

Figure 4. 2-type solution $(\xi_2(t), \eta_2(t))$ of the problem (5.1): a) initial data $\xi_2(0) = 0$, $\eta_2(0) = 137.0835$; b) phase portrait of the difference between neighboring solution $(x(t;\delta),\ y(t;\delta))$ and $(\xi_2(t),\ \eta_2(t))$, $t \in [0,\ 1]$, if $\delta = 0.3$.

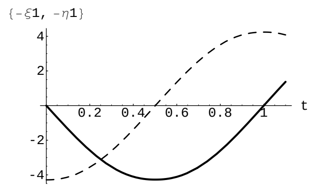


Figure 5. Another 1-type solution of the problem (5.1).

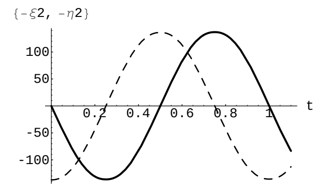


Figure 6. Another 2-type solution of the problem (5.1).

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