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On Construction of Converging Sequences to Solutions of Boundary Value Problems

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Abstract. We consider the Dirichlet problem $x'' = f(t,x)$, $x(a) = A$, $x(b) = B$ under the assumption that there exist the upper and lower functions. We distinguish between two types of solutions, the first one, which can be approximated by monotone sequences of solutions (the so called Jackson–Schrader's solutions) and those solutions of the problem, which cannot be approximated by monotone sequences. We discuss the conditions under which this second type solutions of the Dirichlet problem can be approximated.
Keywords: nonlinear boundary value problems, types of solutions, monotone iterations, multiplicity of solutions, non-monotone iterations.

AMS Subject Classification: 34B15.

1 Introduction

We consider a non-linear second-order differential equation

$$x'' = f(t,x) \tag{1.1}$$

with boundary conditions

$$x(a) = A, \quad x(b) = B, \tag{1.2}$$

where $f(t,x)$ is continuous function along with $f_x(t,x)$ on $[a,b]$.

Geometric interpretation of this problem (1.1)–(1.2) is as follows: it is necessary to find the integral curve passing through two points with coordinates (a,A) and (b,B).

Let us assume that there exist lower α and upper β functions for the problem (1.1)–(1.2). Functions α and β, according to the definition, are such functions that satisfy the following conditions:

$$\alpha \le \beta, \quad \alpha'' \ge f(t,\alpha), \quad \beta'' \le f(t,\beta), \quad \forall t \in [a,b], \tag{1.3}$$
$$\alpha(a) \le A \le \beta(a), \quad \alpha(b) \le B \le \beta(b).$$

Let us mention the result from [8], p. 318, Th.7.20.

Theorem 1. *Assume f is continuous on $[a,b]\times\mathbb{R}$ and $\alpha(t)$, $\beta(t)$ are lower and upper functions of $x''=f(t,x)$, respectively, with $\alpha(t)\le\beta(t)$ on $[a,b]$. If A and B are constants such that $\alpha(a)\le A\le\beta(a)$ and $\alpha(b)\le B\le\beta(b)$, then the BVP* (1.1)–(1.2) *has a solution $x(t)$ satisfying $\alpha(t)\le x(t)\le\beta(t)$, $\quad\forall t\in[a,b]$.*

We suppose that

$$\alpha<\beta,\quad \alpha(a)<A<\beta(a),\quad \alpha(b)<B<\beta(b)\quad t\in[a,b].$$

For solutions of the problem (1.1)–(1.2) there exist schemes of constructing the monotone iterations, they were considered in [1, 2, 3, 4, 9, 10]. For recent works on the subject consult [5, 6] and references therein.

The objective of this work is to introduce non-monotone approximation schemes. They are needed because monotone iterations converge to solutions of definite type, namely, Jackson's–Schrader's type solutions [7] (in our terminology, 0-type solutions). Simple examples show, however, that there are possible also solutions of different oscillatory types. They cannot be approximated by monotone approximations. This new scheme is explained on the particular example. In Section 2 definitions are given. In Section 3 the main result is proved. In Section 4 conclusions are given.

2 Definitions

Second-order BVP and types of solutions. Assume $x(t)$ is a solution of the boundary value problem

$$\begin{cases} x''=f(t,x), & a<x<b,\\ x(a)=A, & x(b)=B. \end{cases}$$

Provided that $f(t,x)$ has a continuous partial derivative f_x we construct the equation of variations and boundary conditions for a particular solution $x(t)$

$$\begin{cases} y''=f_x(t,x(t))y, & a<x<b,\\ y(a)=0, & y'(a)=1. \end{cases}\tag{2.1}$$

Definition 1. We say that $x(t)$ is an i-type solution if $y(t)$ has exactly i zeros in $(a;b)$.

Different types of solutions were considered in [11], but for our purposes it is sufficient to use Definition 1.

The monotone iteration. If the conditions of Theorem 7.20 ([8], p. 318) are satisfied then there exist sequences $\{\overline{x_i}\}$ and $\{\underline{x_i}\}$ such that

$$\overline{x}_1(t)\ge\overline{x}_2(t)\ge\cdots\ge\overline{x}_n(t)\ge\cdots\quad\text{and}\quad \underline{x}_1(t)\le\underline{x}_2(t)\le\cdots\le\underline{x}_n(t)\le\cdots.$$

The upper sequence converges to a maximal solution x_{max} of the problem (1.1)–(1.2) and the lower sequence converges to x_{min}.

Using monotone sequences we take boundary values greater or smaller then the given A and B. *Straight* sequence is defined by auxiliary boundary conditions $x(a) = A_n, x(b) = B_n$, where $A_n > A$, $B_n > B$ (or symmetrically, $A_n < A, B_n < B$). In order to construct non-monotone sequences we take the left value greater then A ($A_n > A$) and the right value smaller then $B(B_n < B)$, or symmetrically, $A_n < A, B_n > B$. We call such sequences *diagonal*. The elements of diagonal sequence have zero points but these points may be different for each solutions. Therefore the problem cannot be reduced to monotone convergences on different intervals.

The usage of diagonal and straight sequences allows us to treat different type solutions.

3 Main result

Theorem 2. *If there exists a sequence $\{x_n\}$, consisting of solutions of the same type (i-type solutions according to Definitions 1) and $\alpha(t) < x_n < \beta(t)$ then there exists a subsequence converging to a similar type solution x of the problem* (1.1)–(1.2). *"Similar" type solution means that either $y(t)$ (a solution of* (2.1) *corresponding to $x(t)$) has exactly i zeros in $(a; b)$ and then $t = b$ can be an $i + 1$-th zero, or $y(t)$ has exactly i zeros in $(a; b]$ and then $t = b$ can be an i-th zero.*

Proof. Let us construct the equations of variations for the solution $x_n(t)$

$$y_n'' = f_x(t, x_n(t))y_n \tag{3.1}$$

and let the following initial conditions

$$y_n(a) = 0, \quad y_n'(a) = 1 \tag{3.2}$$

hold. On the right–hand side of the equation (1.1) function $f(t; x)$ can be unbounded, but since we consider $f(t, x)$ for $t \in [a; b]$ and $x \in [\alpha, \beta]$ the restriction of $f(t, x)$ to the described (t, x)-region is bounded, therefore $|f(t, x)| < M$ for (t, x) considered.

According to Arzela-Ascoli criterium [8], any infinite compact sequence contains a convergent subsequence. In order to use it, compactness of the infinite number of functions $\{x_n\}$ and $\{y_n\}$ within the space C^1 is to be shown. Thus, we have to show that $\{x_n\}$, $\{x_n'\}$, $\{y_n\}$ and $\{y_n'\}$, are equicontinuous and equibounded. Let us show compactness of the sequence $\{x_n(t)\}$ of the solutions of the problem (1.1)-(1.2). First of all, show that the infinite sequence $\{x_n(t)\}$ is bounded. That is because

$$|x_n(t)| < \max\{|\beta(t)|, |\alpha(t)|\}.$$

Let us introduce a constant $K = \max\{|\beta(t)|, |\alpha(t)|\}$. We get that $\forall t \in [a, b]$, $n \in \mathbb{N}$ the sequence $|x_n(t)| < K$ is bounded.

Next we show that the infinite sequence $\{x_n'(t)\}$ is equibounded. First we prove , that for any x_n there exists t_0 such that the following inequality is true:

$$|x_n'(t_0)| < \frac{2K}{b-a}. \tag{3.3}$$

Let us assume that the opposite statement is valid, that is, $\forall t \in [a,b]$ one of the following inequalities is satisfied:

$$x_n'(t) \geq \frac{2K}{b-a} \quad \text{or} \quad x_n'(t) \leq -\frac{2K}{b-a}. \tag{3.4}$$

Integrate both parts of the inequalities (3.4):

$$\int_a^t x_n'(s)\,ds \geq \frac{2K}{b-a}\int_a^t ds, \quad \text{or} \quad \int_a^t x_n'(s)\,ds \leq -\frac{2K}{b-a}\int_a^t ds.$$

As a result, we get:

$$x_n(t) - x_n(a) \geq \frac{2K}{b-a}(t-a), \quad \text{or} \quad x_n(t) - x_n(a) \leq -\frac{2K}{b-a}(t-a). \tag{3.5}$$

In inequalities (3.5) at $t = b$, we get:

$$x_n(b) - x_n(a) \geq 2K \quad \text{or} \quad x_n(b) - x_n(a) \leq -2K. \tag{3.6}$$

The last two relations contradict the choice of the number K, thus there exists $t_0 \in [a,b]$, for which inequalities (3.3) are valid. Before to evaluate $\{x_n'(t)\}$ in modulus, write

$$x_n'(t) = x_n'(t_0) + \int_{t_0}^t x_n''(s)\,ds,$$

where $a < t_0 < b$. Then,

$$\begin{aligned} \big|x_n'(t)\big| = \Big|x_n'(t_0) + \int_{t_0}^t x_n''(s)\,ds\Big| &< \frac{2K}{b-a} + \Big|\int_{t_0}^t f(s, x_n(s))\,ds\Big| \\ &< \frac{2K}{b-a} + M(b-a). \end{aligned} \tag{3.7}$$

Thus, it is shown that the infinite sequence $\{x_n'(t)\}$ is uniformly bounded.

It can be shown now that the sequence is equicontinuous. First of all, show this feature for the infinite number of functions $\{x_n(t)\}$. According to the definition of equicntinuity $\forall \varepsilon > 0\ \exists \delta > 0$, such that as soon as $|t_2 - t_1| < \delta$ then it follows that $|x_n(t_1) - x_n(t_2)| < \varepsilon$ for any n. One has, according to Lagrange's Mean Value Theorem, that $\forall t_1, t_2 \in [a,b]$

$$\frac{x_n(t_1) - x_n(t_2)}{t_1 - t_2} = x_n'(\xi), \quad t_1 < \xi < t_2.$$

We can evaluate the modulus of the difference, using (3.7).

$$\big|x_n(t_1) - x_n(t_2)\big| = \big|x_n'(\xi)\big|\big|t_1 - t_2\big| < \Big(\frac{2K}{b-a} + M(b-a)\Big)\big|t_1 - t_2\big|.$$

As a result, we get the required value $\delta > 0$:

$$\delta = \frac{\varepsilon}{2K/(b-a) + M(b-a)}.$$

Now, let us evaluate the modulus of the difference $|x'_n(t_1) - x'_n(t_2)|$ and find the corresponding δ. Using Lagrange's Mean Value Theorem, it is possible to state:

$$|x'_n(t_1) - x'_n(t_2)| = |x''_n(\eta)||t_1 - t_2|, \quad t_1 < \eta < t_2, \ \forall t_1, t_2 \in [a, b].$$

Using the condition of problem (1.1), in the last expression change $x''_n(\eta)$ to $f(\eta, x_n(\eta))$, and then apply the fact that function f is bounded within the interval $[a, b]$:

$$\big|x'_n(t_1) - x'_n(t_2)\big| = \big|f(\eta, x_n(\eta))\big|\big|t_1 - t_2\big| < M\big|t_1 - t_2\big|.$$

As a result of this analysis we get that $\delta = \varepsilon/M$.

We wish to show now that $\{y_n(t)\}$ contains a converging to $y(t)$ subsequence. Let us prove first that the sequences $\{y_n(t)\}$ and $\{y'_n(t)\}$ are uniformly bounded. Denote $f_x(t, x_n(t)) = \varphi(t)$. It is clear that $|\varphi(t)| < M_1 = const$ because $a \le t \le b$ and $\alpha < x_n < \beta$. We want to show that there exists $N = const$ such that $|y_n(t)| < N$ for all $t \in [a, b]$.

In order to show that the limiting function $x(t)$ possesses the property described in the statement of Theorem 2 let us write equation (3.1) as a system

$$\begin{cases} y' = u, \\ u' = \varphi(t)y. \end{cases} \tag{3.8}$$

and introduce polar coordinates

$$\begin{cases} u' = \rho' \cos\theta - \rho \sin\theta, \\ y' = \rho' \sin\theta + \rho\cos\theta, \end{cases} \tag{3.9}$$

where $u = \rho\cos\theta$, $y = \rho\sin\theta$. Using the (3.9) we get

$$\rho' = \frac{\begin{vmatrix} u' & -\rho\sin\theta \\ y' & \rho\cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix}} = \frac{u'\rho\cos\theta + y'\rho\sin\theta}{\rho\cos^2\theta + \rho\sin^2\theta} = u'\cos\theta + y'\sin\theta.$$

Using the system (3.8) we get

$$\begin{aligned} \rho' &= \varphi(t)y\cos\theta + u\sin\theta = \varphi(t)\rho\sin\theta\cos\theta + \rho\cos\theta\sin\theta \\ &= \rho\big[\varphi(t) + 1\big]\sin\theta\cos\theta = \frac{1}{2}\rho\big[\varphi(t) + 1\big]\sin 2\theta. \end{aligned}$$

According to the initial conditions (3.2) we have that $\rho(0) = 1$. It follows that $\rho(t) = \rho(0)e^{\int R(t)\,dt}$, where

$$R(t) = 0.5(\varphi(t) + 1)\sin 2\theta \le (1 + M_1)/2.$$

Now we can evaluate $|y(t)|$:

$$|y(t)| = |\rho(t)\sin\theta| < \rho(0)e^{(1+M_1)(b-a)/2}.$$

As a result, get the value of the constant $N = e^{(1+M_1)(b-a)/2}$. Therefore, $|y_n(t)| < N$. The sequence $\{y'_n(t)\}$ is bounded because

$$y' = \rho'\sin\theta + \rho\cos\theta.$$

Now let us evaluate the modulus of the difference $|y_n(t_1) - y_n(t_2)|$ and find the corresponding δ. For this, we write

$$y_n(t) = y_n(0) + \int_0^t y'_n(s)\,ds,$$
$$|y_n(t_1) - y_n(t_2)| = \left|\int_{t_1}^{t_2} y'_n(s)\,ds\right| < P|t_2 - t_1|,$$

where $|y'_n(t)| < P = const\ \forall t \in [a,b]$. As a result, we get the value $\delta = \varepsilon/P$. Consider the sequence $\{y'_n(t)\}$ and evaluate the modulus of the difference

$$|y'_n(t_1) - y'_n(t_2)| = \left|\int_{t_1}^{t_2} y''_n(s)\,ds\right| = \left|\int_{t_1}^{t_2} f_x(s, x_n(s))y_n(s)\,ds\right|$$
$$\leq M_1\int_{t_1}^{t_2} y_n(s)\,ds \leq M_1 N|t_2 - t_1| \tag{3.10}$$

As the result, we get that: $\delta = \varepsilon/NM_1$. There are sequences $\{x_n(t)\}$, $\{y_n(t)\}$ and limiting functions $x(t)$, $y(t)$. It follows from the hypotheses of the theorem that a polar function $\Theta_n(t)$ corresponding to $\{y_n(t)\}$ fulfils the condition $\pi(i+1) \geq \Theta_n(b) > \pi \cdot i$. This is the same as $y_n(t)$ would have exactly i zeros in the interval, and $y_n(b) \neq 0$. Then the limiting function $y(t)$ will fulfil the inequalities $\pi(i+1) \geq \Theta(b) \geq \pi i$, and this is the same that either $x(t)$ has exactly i zeros in $(a,b]$ or $x(t)$ has exactly i zeros in (a,b) and $(i+1)$-th zero at $t = b$. The proof is complete. □

Next we consider one example.

Example 1.

$$x'' = x^3 - k^2 x, \quad 0 < t < 1, \;\; k = 5\pi/2, \tag{3.11}$$
$$x(0) = 0, \quad x(1) = 0. \tag{3.12}$$

The upper and lower functions are defined as $\beta(t) = 8$ and $\alpha(t) = -8$, then all conditions (1.3) are satisfied. The solutions of this problem are shown in Fig. 1.

In Fig. 2 and Fig. 3 we see that there are two subsequences of straight sequences, but they converge to different solutions, i.e., to 0-solution and to 2-solution. The second subsequence is not monotone. What is the difference between these two subsequences? To answer this question, we consider a phase plane. We show monotone iterations on a phase plane (see, Fig. 4).

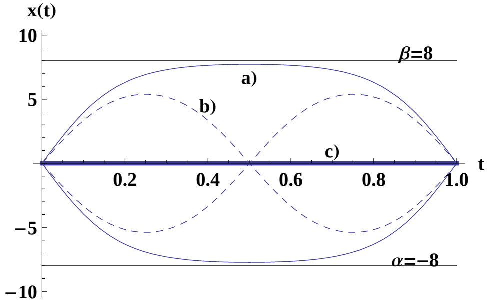

Figure 1. All solutions of the problem (3.11)–(3.12): a) the 0-type solutions; b) the 1-type solutions; c) the 2-type solution (the unique one).

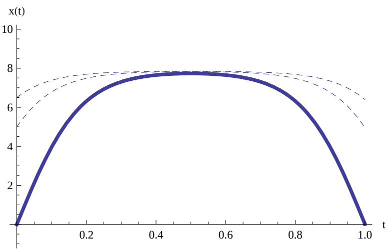

Figure 2. Some approximations of 0-type solution.

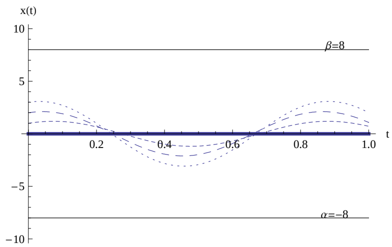

Figure 3. Some approximations of 2-type solution.

We construct also the diagonal sequence. It appears that elements of this sequence have zeros at some points. The interval $[a, b]$ generally cannot be decomposed to subintervals in which convergence is monotone since zeros of $x_n(t)$ generally do not coincide. The diagonal sequence converges to a solution also, and this convergence is not monotone by construction. We have α and β functions. We have 0-type solutions one of them is depicted in Figure 2. We have also 1-type solutions which are shown in Figure 5. 1-type solutions are approximated by non-monotone diagonal sequence. Finally there is 2-type solution (the trivial one) which can be approximated by straight non-monotone iterations (see, Fig. 6)

4 Conclusions

- We show that solutions of non-zero-type cannot be approximated by monotone sequences. They can be approximated by suitable straight or diagonal sequences. It follows that straight and diagonal sequences of solutions can be constructed.
- Auxiliary boundary value problems, which contain elements of straight and diagonal sequences, can have multiple solutions.
- These solutions can be arranged in sequences of similar type solutions. These sequences converge to the solution of the same type. We note,

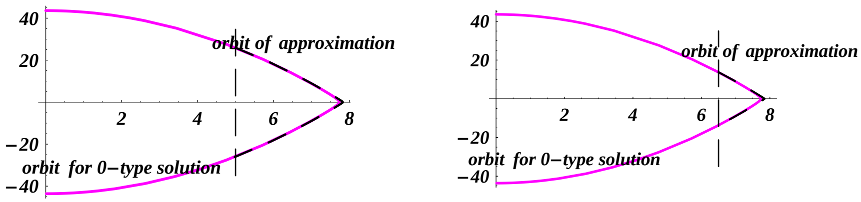

Figure 4. The phase portrait of the 0-type solution (solid) and two approximations (dashed).

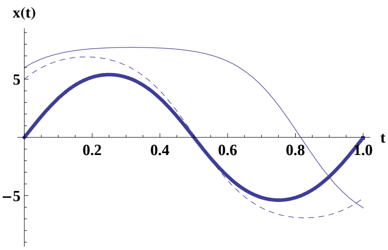

Figure 5. Some approximations of 1-type solution.

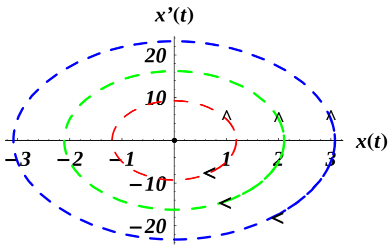

Figure 6. The phase portrait of the 2-type solution (a point at the origin) and approximations.

that these sequences also contain subsequences converging to solutions of different type for a given BVP.

- If appears that solutions of different types can coexist and a converging sequence should be selected carefully in order to ensure convergence to a definite type solution.

References

[1] H. Amann. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.*, **18**:620–709, 1976. Doi:10.1137/1018114.

[2] A. Cabada and I.J. Nieto. A generalization of the monotone – iterative technique for nonlinear second order periodic boundary value problems. *J. Mathemat. Anal. and Appl.*, **151**:181–189, 1999. Doi:10.1016/0022-247X(90)90249-F.

[3] M. Cherpion, C. De Coster and P. Habets. Monotone iterative methods for boundary value problems. *Diff. and Integral Equat.*, **12**:309–338, 1999.

[4] C.De Coster and P. Habets. *Two-point boundary value problems: lower and upper solutions.* Elsevier, 2006.

[5] M. Dobkevich and F. Sadyrbaev. Types of solutions and approximation of solutions of second order nonlinear boundary value problems. In *Amer. Inst. Phys. Conference Proceedings. Numerical Analysis and applied mathematics. International Conference on Numerical Analysis and Applied Mathematics 2009.*

Rethymno, Crete, Greece, September 18–22 2009, volume 1168, pp. 260–263, 2009.

[6] M.K. Grammatikopoulos and P.S. Kelevedjiev. Minimal and maximal solutions for two-point boundary-value problems. *Electronic Journal of Differential Equations*, **2003**(21):1–14, 2003.

[7] L.K. Jackson and K.W. Schrader. Comparison theorems for nonlinear differential equations. *Journal of Differential Equations*, **3**:248–255, 1967. Doi:10.1016/0022-0396(67)90029-0.

[8] W. Kelly and A. Peterson. *The Theory of Differential Equations: Classical and Qualitative*. Pearson Education, New Jersy, 2004.

[9] P. Korman. Monotone approximations of unstable solutions. *J. Comput. and Appl. Math.*, **136**:309–315, 2001. Doi:10.1016/S0377-0427(00)00622-1.

[10] V. Lakshmikantham, G. Ladde and A. Vatsala. *Monotone Iterative Techniques in Nonlinear Differential Equations*. Pitman Publishing, London, 1985.

[11] I. Yermachenko and F. Sadyrbaev. Quasilinearization and multiple solutions of the Emden–Fowler type equation. *Math. Model. Anal.*, **10**(1):41–50, 2005.