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On Approximation of Value Functions for Controlled Discontinuous Random Processes

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Abstract. We consider the problem of approximation of value functions for controlled possibly degenerated diffusion processes with jumps by using piece-wise constant control policies. A rate of convergence for the corresponding value functions is established provided that the coefficients of controlled processes are sufficiently smooth. The paper extends the results of N.V. Krylov to a more general class of controlled processes.

Keywords: Controlled diffusion processes with jumps, Bellman principle, piece-wise constant strategies, value functions.

AMS Subject Classification: 60J75.

1 Introduction

The paper is devoted to numerical approximations in the control theory of random processes. Recall that under suitable assumptions such value functions are probabilistic solutions to fully nonlinear parabolic integro-differential Bellman equations of the second order (see [3]). In the paper, the analysis of approximations of value functions is based on the same ideas as that for value functions of controlled diffusion processes considered by N.V. Krylov in [2].

The paper is organised as follows. The main results are given in Section 2. Section 3 contains auxiliary results. In Section 4, the proofs of the main results are presented.

Throughout the paper R^d is a d-dimensional Euclidean space, A is a separable metric space, $T \in (0,\infty)$, $K \in [1,\infty)$, $\delta_0 \in (0,1]$ and $\delta \in (0,1]$ are some fixed constants. By N we denote various constants depending only on T, K, d and d_1, where d_1 is introduced in the next section.

2 Main Results

Let Π be a σ-finite nonnegative measure on $(R^m, \mathcal{B}(R^m))$ such that $\Pi(z\colon |z| > \varepsilon) < \infty$ for any $\varepsilon > 0$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration of σ-algebras $\mathbb{F} = (\mathcal{F}_t,\ t \geq 0)$ satisfying the usual conditions. Assume that on this probability space a d_1-dimensional $\mathbb{F}$-adapted Wiener process W_t, $t \geq 0$, and $\mathbb{F}$-adapted Poisson random measure $p(dt, dz)$ on $([0,\infty) \times R^m, \mathcal{B}([0,\infty)) \otimes \mathcal{B}(R^m))$ with a compensator $\Pi(dz)dt$ are given. Let

$$q(dt, dz) = p(dt, dz) - \Pi(dz)dt$$

be a martingale measure.

Definition 1. An A-valued random process $\alpha_t = \alpha_t(\omega)$, $t \geq 0$, $\omega \in \Omega$, is called $\mathbb{F}$-admissible if it is $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable and α_t is $\mathcal{F}_t$-measurable for each $t \geq 0$.

The set of all $\mathbb{F}$-admissible processes is denoted by $\mathfrak{A}$. Let $\mathfrak{A}_h$ be the subset of $\mathfrak{A}$ consisting of all processes α_t which are constant on intervals $[0, h^2)$, $[h^2, 2h^2)$, etc.

Fix integers $d \geq 1$, $d_1 \geq 1$, $m \geq 1$ and assume that we have the following functions: a $d \times d_1$ matrix-valued $\sigma(\alpha, t, x)$ and R^d-valued $b(\alpha, t, x)$ on $A \times [0,\infty) \times R^d$, a R^d-valued $c(\alpha, t, x, z)$ on $A \times [0,\infty) \times R^d \times R^m$ and real-valued $g(x)$ on R^d. We assume that these functions are Borel measurable.

For any matrix $\sigma = (\sigma_{ij})$ and function $c : R^m \to R^d$ denote

$$\|\sigma\| = \Big\{\sum_{i,j} \sigma_{ij}^2\Big\}^{1/2}, \quad \|c\|_{2,\Pi} = \Big\{\int_{R^m} |c(z)|^2 \Pi(dz)\Big\}^{1/2}.$$

Further we shall use the following assumption.

Assumption 1. (i) The functions σ and b are continuous with respect to α and for each $\alpha \in A$, $t \in [0,\infty)$, $x \in R^d$ we have

$$\|c(\alpha_n, t, x, \cdot) - c(\alpha, t, x, \cdot)\|_{2,\Pi} \to 0 \ \text{ as } \ \alpha_n \to \alpha;$$

(ii) For each $\alpha \in A$, $t \in [0,\infty)$, $x, y \in R^d$

$$\begin{aligned} &\|\sigma(\alpha, t, x)\| + |b(\alpha, t, x)| + \|c(\alpha, t, x, \cdot)\|_{2,\Pi} \leq K, \\ &\|\sigma(\alpha, t, x) - \sigma(\alpha, t, y)\| + |b(\alpha, t, x) - b(\alpha, t, y)| \\ &\qquad + \|c(\alpha, t, x, \cdot) - c(\alpha, t, y, \cdot)\|_{2,\Pi} \leq K|x - y|; \end{aligned}$$

(iii) For each $\alpha \in A$, $s, t \in [0,\infty)$, $x \in R^d$

$$\begin{aligned} &\|\sigma(\alpha, t, x) - \sigma(\alpha, s, x)\| + |b(\alpha, t, x) - b(\alpha, s, x)| \\ &\qquad + \|c(\alpha, t, x, \cdot) - c(\alpha, s, x, \cdot)\|_{2,\Pi} \leq K|t - s|^{\delta_0/2}; \end{aligned}$$

(iv) For each $x, y \in R^d$

$$|g(x)| \leq K, \quad |g(x) - g(y)| \leq K|x - y|^{\delta}.$$

Let Assumption 1 (ii) be satisfied. Then, by Itô's theorem, for each $\alpha \in \mathfrak{A}$, $s \in [0,T]$ and $x \in R^d$ there exists a unique solution $X_t = X_t^{\alpha,s,x}$, $t \geq 0$, of the equation

$$X_t = x + \int_0^t \sigma(\alpha_r, s+r, X_r)\, dW_r + \int_0^t b(\alpha_r, s+r, X_r)\, dr + \int_0^t \int c(\alpha_r, s+r, X_r, z)\, q(dr, dz). \tag{2.1}$$

For $\alpha \in \mathfrak{A}$, $s \in [0,T]$ and $x \in R^d$ define

$$v^\alpha(s,x) = \mathbb{E} g(X_{T-s}^{\alpha,s,x}), \quad v(s,x) = \sup_{\alpha \in \mathfrak{A}} v^\alpha(s,x), \quad v_h(s,x) = \sup_{\alpha \in \mathfrak{A}_h} v^\alpha(s,x).$$

Theorem 1. *Let Assumption* 1 *be satisfied. Then for each* $s \in [0,T]$, $x \in R^d$, $h \in (0,1)$

$$|v(s,x) - v_h(s,x)| \leq N h^{\varkappa},$$

where $\varkappa = \delta\delta_0^2/(2 + \delta\delta_0 + \delta_0 - \delta)$ *and the constant* N *depends only on* T, K, d *and* d_1.

Remark 1. The largest value of $\varkappa$ for $\delta_0 \in (0,1]$, $\delta \in (0,1]$ is equal to $1/3$ and is achieved with $\delta_0 = \delta = 1$.

Theorem 1 can be used to reduce the problem of calculation of value function v to that of value function v_h for piece-wise constant control policies.

For $\alpha \in A$, $0 \leq s \leq t < \infty$, $x \in R^d$ and any Borel measurable bounded real-valued function f on R^d define

$$G_{s,t}^\alpha f(x) = \mathbb{E} f(X_{t-s}^{\alpha,s,x}), \quad G_{s,t} f(x) = \sup_{\alpha \in A} G_{s,t}^\alpha f(x).$$

According to the dynamic programming principle (see Lemma 1 in the next section) for $s + h^2 \leq T$

$$v_h(s,x) = G_{s,s+h^2} v_h(s+h^2, \cdot)(x).$$

Therefore $v_h(s,x)$ can be found from its boundary value $v_h(T,\cdot) = g(\cdot)$ by backward iteration.

In order to simplify the calculation one can apply the Euler scheme as the simplest approximation of controlled process. For $\alpha \in A$, $s,t \geq 0$, $x \in R^d$ define

$$Y_t^{\alpha,s,x} = x + \sigma(\alpha,s,x) W_t + b(\alpha,s,x)t + \int_0^t \int c(\alpha,s,x,z)\, q(dr,dz)$$

and recursively

$$\overline{v}_h(s,x) = g(x) \quad \text{if} \;\; s \in (T-h^2, T],$$
$$\overline{v}_h(s,x) = \overline{G}_{s,s+h^2} \overline{v}_h(s+h^2,\cdot)(x) \quad \text{if} \;\; s \leq T-h^2,$$

where

$$\overline{G}_{s,s+t}^\alpha f(x) = \mathbb{E} f(Y_t^{\alpha,s,x}), \quad \overline{G}_{s,s+t} f(x) = \sup_{\alpha \in A} \overline{G}_{s,s+t}^\alpha f(x).$$

Theorem 2. *Let Assumption* 1 *be satisfied. Then for each* $s \in [0,T]$, $x \in R^d$ *and* $h \in (0,1)$

$$|v(s,x) - \overline{v}_h(s,x)| \le N h^{\varkappa},$$

$\varkappa = \delta\delta_0^2/(2+\delta\delta_0+\delta_0-\delta)$ *and the constant* N *depends only on* T, K, d *and* d_1.

3 Auxiliary Results

The following lemma states the dynamic programming principle for the value functions v and v_h.

Lemma 1. *Let Assumption* 1 *be satisfied. Then*:

(i) *for each* $x \in R^d$ *and* $0 \le s \le t \le T$

$$v(s,x) = \sup_{\alpha \in \mathfrak{A}} \mathbb{E} v(t, X_{t-s}^{\alpha,s,x});$$

(ii) *for each* $x \in R^d$ *and* $0 \le s \le t \le T$ *such that* $(t-s)/h^2$ *is an integer*

$$v_h(s,x) = \sup_{\alpha \in \mathfrak{A}_h} \mathbb{E} v_h(t, X_{t-s}^{\alpha,s,x}).$$

The proof of the lemma is similar to that for controlled diffusion processes (see Theorem 3.1.6, Exercise 3.2.1 and Lemma 3.3.1 in [1]) and therefore it is omitted here.

In order to prove the main results of the paper, we introduce an auxiliary controlled process and value functions u and u_h as follows. Let

$$B_1 = \{x \in R^d\colon\ |x| < 1\}, \quad B = A \times \{(r,\xi)\colon r \in (-1,0),\ \xi \in B_1\}.$$

We extend the functions σ, b, c for $t < 0$ by

$$\sigma(\alpha,t,x) = \sigma(\alpha,0,x), \quad b(\alpha,t,x) = b(\alpha,0,x), \quad c(\alpha,t,x,z) = c(\alpha,0,x,z)$$

and for a fixed $\varepsilon \in (0,1)$ and each $\beta = (\alpha,r,\xi) \in B$, $t \in R$, $x \in R^d$, $z \in R^m$ define

$$\sigma(\beta,t,x) = \sigma(\alpha, t+\varepsilon^2 r, x+\varepsilon\xi), \quad b(\beta,t,x) = b(\alpha, t+\varepsilon^2 r, x+\varepsilon\xi),$$
$$c(\beta,t,x,z) = c(\alpha, t+\varepsilon^2 r, x+\varepsilon\xi, z).$$

Let $\mathfrak{B}$ be the set of all measurable $\mathbb{F}$-adapted B-valued processes and $\mathbb{B}_h$ be the subset of $\mathfrak{B}$ consisting of functions which are constant on intervals $[0,h^2)$, $[h^2,2h^2)$, etc. Finally, for each $\beta \in \mathfrak{B}$, $s \in [0,S]$, $x \in R^d$, we define a controlled jump-diffusion process $X_t^{\beta,s,x}$, $t \ge 0$, as a solution to (2.1) with α_r replaced by β_r and the value functions

$$u^\beta(s,x) = \mathbb{E} g(X_{S-s}^{\beta,s,x}), \quad u(s,x) = \sup_{\beta \in \mathfrak{B}} u^\beta(s,x), \quad u_h(s,x) = \sup_{\beta \in \mathfrak{B}_h} u^\beta(s,x),$$

where $S = T + \varepsilon^2$. Obviously, the process $X_t^{\beta,s,x}$ and value functions u, u_h depend also on ε what is not explicitly shown just for brevity of notation.

Remark 2. Since the controlled process $X_t^{\beta,s,x}$ and the value functions u, u_h are defined in the same way as $X_t^{\alpha,s,x}$, v and v_h, Lemma 1 implies that for each $x \in R^d$ and $0 \le s \le t \le S$

$$u(s,x) = \sup_{\beta \in \mathfrak{B}} \mathbb{E} u(t, X_{t-s}^{\beta,s,x})$$

and for each $x \in R^d$ and $0 \le s \le t \le S$ such that $(t-s)/h^2$ is an integer

$$u_h(s,x) = \sup_{\beta \in \mathfrak{B}_h} \mathbb{E} u_h(t, X_{t-s}^{\beta,s,x}).$$

Let $y_t^{(i)}$, $t \ge 0$, $i = 1,2$, be d-dimensional solutions to the equations

$$y_t = x^{(i)} + \int_0^t \sigma^{(i)}(s, y_s)\, dW_s + \int_0^t b^{(i)}(s, y_s)\, ds + \int_0^t \int c^{(i)}(s, y_s, z)\, q(ds, dz),$$

respectively, the coefficients of which are measurable and $\mathbb{F}$-adapted random functions satisfying the linear growth and Lipschitz conditions, i.e. for each $t \ge 0$, $x, y \in R^d$, $\omega \in \Omega$ and $i = 1,2$

$$\|\sigma^{(i)}(t,x)\| + |b^{(i)}(t,x)| + \|c^{(i)}(t,x,\cdot)\|_{2,\Pi} \le K(1+|x|),$$
$$\|\sigma^{(i)}(t,x) - \sigma^{(i)}(t,y)\| + |b^{(i)}(t,x) - b^{(i)}(t,y)| + \|c^{(i)}(t,x,\cdot) - c^{(i)}(t,y,\cdot)\|_{2,\Pi} \le K|x-y|).$$

Lemma 2. *Let the above assumptions be satisfied and $T \in (0,\infty)$. Then there is a constant $N = N(K,T)$ such that*

$$\mathbb{E} \sup_{t \le T} |y_t^{(1)} - y_t^{(2)}|^2 \le N(|x^{(1)} - x^{(2)}|^2 + \mathbb{E} \int_0^T (\|\sigma^{(1)}(s, y_s^{(2)}) - \sigma^{(2)}(s, y_s^{(2)})\| + |b^{(1)}(s, y_s^{(2)}) - b^{(2)}(s, y_s^{(2)})|^2 + \|c^{(1)}(s, y_s^{(2)}, \cdot) - c^{(2)}(s, y_s^{(2)}, \cdot)\|_{2,\Pi}^2)\, ds.$$

Proof. Using well-known properties of stochastic integrals, we have

$$\mathbb{E} \sup_{t \le T} |y_t^{(1)} - y_t^{(2)}|^2 \le N(|x^{(1)} - x^{(2)}|^2 + \mathbb{E} \int_0^T (\|\sigma^{(1)}(s, y_s^{(1)}) - \sigma^{(2)}(s, y_s^{(2)})\|^2 + |b^{(1)}(s, y_s^{(1)}) - b^{(2)}(s, y_s^{(2)})|^2 + \|c^{(1)}(s, y_s^{(1)}, \cdot) - c^{(2)}(s, y_s^{(2)}, \cdot)\|_{2,\Pi}^2)\, ds, \tag{3.1}$$

where the constant N depends only on K and T. According to our assumptions,

$$\|\sigma^{(1)}(s, y_s^{(1)}) - \sigma^{(2)}(s, y_s^{(2)})\| \le K|y_s^{(1)} - y_s^{(2)}| + \|\sigma^{(1)}(s, y_s^{(2)}) - \sigma^{(2)}(s, y_s^{(2)})\|$$

and similar inequalities hold for other terms on the right-hand side of (3.1). Therefore, the assertion of the lemma follows by Gronwall's inequality. □

Lemma 3. *Let Assumption 1 be satisfied, $s, s' \in [0,S]$, $x, y \in R^d$ and $\beta = (\alpha, r, \xi) \in \mathfrak{B}$. Then there is a constant $N = N(T,K)$ such that*

$$\mathbb{E} \sup_{t \le S} |X_t^{\beta,s,x} - X_t^{\alpha,s,x}|^2 \le N\varepsilon^{2\delta_0}, \quad \mathbb{E} \sup_{t \le S} |X_t^{\beta,s,x} - X_t^{\beta,s,y}|^2 \le N|x-y|^2,$$
$$\mathbb{E} \sup_{t \le S} |X_t^{\beta,s,x} - X_t^{\beta,s',x}|^2 \le N|s-s'|^{\delta_0},$$

Proof. The estimates of the lemma follow easily from Lemma 2 and Assumption 1. For example,

$$\mathbb{E}\sup_{t\le S}|X_t^{\beta,s,x}-X_t^{\alpha,s,x}|^2 \le N\sup(\|\sigma(\alpha,t+\varepsilon^2 r,x+\varepsilon\xi)-\sigma(\alpha,t,x)\|^2 + |b(\alpha,t+\varepsilon^2 r,x+\varepsilon\xi)-b(\alpha,t,x)|^2 + \|c(\alpha,t+\varepsilon^2 r,x+\varepsilon\xi,\cdot)-c(\alpha,t,x,\cdot)\|^2_{2,\Pi}) \le N\varepsilon^{2\delta_0},$$

where $N=N(T,K)$ and sup is taken over $\alpha\in A$, $t\le S$, $r\in(-1,0)$, $x\in R^d$, $\xi\in B_1$. □

Remark 3. Lemma 2 and Hölder's inequality imply that for each $\alpha\in\mathfrak{A}$, $s,t\ge 0$, $x\in R^d$ and $\mu\in(0,2]$

$$\mathbb{E}|X_t^{\alpha,s,x}-x|^\mu \le \{\mathbb{E}X_t^{\alpha,s,x}-x|^2\}^{\frac{\mu}{2}} \le Nt^{\frac{\mu}{2}},$$

where the constant $N=N(K)$.

Lemma 4. *Let Assumption 1 be satisfied. Then for each $s\in[0,T]$, $x\in R^d$ and $\beta=(\alpha,r,\xi)\in\mathfrak{B}$*

$$|u^\beta(s,x)-v^\alpha(s,x)|\le N\varepsilon^{\delta_0\delta},\quad |u(s,x)-v(s,x)|\le N\varepsilon^{\delta_0\delta},$$
$$|u_h(s,x)-v_h(s,x)|\le N\varepsilon^{\delta_0\delta},$$

where the constant $N=N(T,K)$.

Proof. As can be easily seen, it suffices to prove the first inequality. By Assumption 1, Hölder's inequality and Lemma 3,

$$\begin{aligned}|u^\beta(s,x)-v^\alpha(s,x)| &\le \mathbb{E}|g(X_{S-s}^{\beta,s,x})-g(X_{T-s}^{\alpha,s,x})|\\ &\le K\mathbb{E}|X_{S-s}^{\beta,s,x}-X_{T-s}^{\alpha,s,x}|^\delta \le K\{\mathbb{E}|X_{S-s}^{\beta,s,x}-X_{T-s}^{\alpha,s,x}|^2\}^{\delta/2}\\ &\le 2K\{\mathbb{E}|X_{S-s}^{\beta,s,x}-X_{T-s}^{\beta,s,x}|^2+\mathbb{E}|X_{T-s}^{\beta,s,x}-X_{T-s}^{\alpha,s,x}|^2\}^{\delta/2}\\ &\le N(\varepsilon^2+\varepsilon^{2\delta_0})^{\delta/2}\le N\varepsilon^{\delta_0\delta}.\end{aligned}$$ □

Lemma 5. *Let Assumption 1 be satisfied. Then there is a constant $N=N(T,K)$ such that:*

(i) *for each $s\in[0,S]$ and $x,y\in R^d$*

$$|u(s,x)-u(s,y)|\le N|x-y|^\delta,\quad |u_h(s,x)-u_h(s,y)|\le N|x-y|^\delta;$$

(ii) *for each $0\le s\le t\le S$ and $x\in R^d$*

$$|u(s,x)-u(t,x)|\le N|t-s|^{\delta/2},\quad |u_h(s,x)-u(t,x)|\le N(h^{\delta_0\delta}+|t-s|^{\delta/2}).$$

Proof. (i) By Assumption 1, Hölder's inequality and Lemma 3,

$$\begin{aligned}|u(s,x)-u(s,y)| &\le \sup_{\beta\in\mathfrak{B}}\mathbb{E}|g(X_{S-s}^{\beta,s,x})-g(X_{S-s}^{\beta,s,y})| \le K\sup_{\beta\in\mathfrak{B}}\mathbb{E}|X_{S-s}^{\beta,s,x}-X_{S-s}^{\beta,s,y}|^\delta\\ &\le K\sup_{\beta\in\mathfrak{B}}\{\mathbb{E}|X_{S-s}^{\beta,s,x}-X_{S-s}^{\beta,s,y}|^2\}^{\delta/2}\le N|x-y|^\delta.\end{aligned}$$

The same arguments prove the second assertion in (i).

(ii) By Remark 2, the assertion (i), Hölder's inequality and Lemma 2,

$$|u(s,x)-u(t,x)| = |\sup_{\beta\in\mathfrak{B}} \mathbb{E}u(t,X^{\beta,s,x}_{t-s}) - u(t,x)| \le N \sup_{\beta\in\mathfrak{B}} \mathbb{E}|X^{\beta,s,x}_{t-s}-x|^{\delta}$$
$$\le N\sup_{\beta\in\mathfrak{B}}\{\mathbb{E}|X^{\beta,s,x}_{t-s}-x|^2\}^{\delta/2} \le N|t-s|^{\delta/2}.$$

Similarly, if $(t-s)/h^2$ is an integer,

$$|u_h(s,x)-u_h(t,x)| \le N|t-s|^{\delta/2}. \tag{3.2}$$

On the other hand, by Hölder's inequality and Lemmas 2, 3, for each $0 \le s \le t \le S$ and $x \in R^d$

$$|u_h(s,x)-u_h(t,x)| \le \sup_{\beta\in\mathfrak{B}_h} \mathbb{E}|g(X^{\beta,s,x}_{S-s}) - g(X^{\beta,t,x}_{S-t})|$$
$$\le K \sup_{\beta\in\mathfrak{B}_h} \mathbb{E}|X^{\beta,s,x}_{S-s} - X^{\beta,t,x}_{S-t}|^{\delta} \le K \sup_{\beta\in\mathfrak{B}_h}\{\mathbb{E}|X^{\beta,s,x}_{S-s} - X^{\beta,t,x}_{S-t}|^2\}^{\delta/2}$$
$$\le 2K \sup_{\beta\in\mathfrak{B}_h}\{\mathbb{E}|X^{\beta,s,x}_{S-s} - X^{\beta,s,x}_{S-t}|^2 + \mathbb{E}|X^{\beta,s,x}_{S-t} - X^{\beta,t,y}_{S-t}|^2\}^{\delta/2}$$
$$\le N(|t-s| + |t-s|^{\delta_0})^{\delta/2} \le N|t-s|^{\delta_0\delta/2}. \tag{3.3}$$

The inequalities (3.2) and (3.3) imply the second assertion in (ii). The lemma is proved. □

For $\alpha \in A$ and smooth functions $f : [0,\infty)\times R^d \to R$ introduce the operator

$$L^{\alpha}f(t,x) = \frac{\partial f}{\partial t}(t,x) + \sum_{i,j=1}^{d} a_{ij}(\alpha,t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x)$$
$$+ \sum_{i=1}^{d} b_i(\alpha,t,x)\frac{\partial f}{\partial x_i}(t,x) + \int_{R^m} \nabla^2_{c(\alpha,t,x,z)} f(t,x)\Pi(dz),$$

where $a = \frac{1}{2}\sigma\sigma^*$ and

$$\nabla^2_y f(t,x) = f(t,x+y) - f(t,x) - \sum_{i=1}^{d}\frac{\partial f}{\partial x_i}(t,x)y_i.$$

Let a nonnegative function $\zeta \in C_0^{\infty}((-1,0)\times B_1)$ be such that

$$\int_{-1}^{0}\int_{B_1}\zeta(t,x)\,dt\,dx = 1.$$

For $\varepsilon > 0$ define $\zeta_\varepsilon(t,x) = \varepsilon^{-d-2}\zeta(t/\varepsilon^2, x/\varepsilon)$. Further we use the following notation:

$$f^{(\varepsilon)}(t,x) = \int_{R^{d+1}} \zeta_\varepsilon(t-s,x-y)f(s,y)\,ds\,dy, \quad |f|_0 = \sup_{\substack{t\in[0,T]\\ x\in R^d}} |f(t,x)|,$$

$$[f]_{\frac{\varkappa}{2},\varkappa} = \sup_{\substack{s,t\in[0,T]\\ x,y\in R^d}} \frac{|f(s,x)-f(t,y)|}{|s-t|^{\varkappa/2} + |x-y|^{\varkappa}}, \quad \varkappa \in (0,1].$$

Lemma 6. *Let Assumption 1 be satisfied and $\varepsilon \geq h^{\delta_0}$. Then for each $\alpha \in A$*

$$[L^\alpha u_h^{(\varepsilon)}]_{\frac{\delta_0}{2},\delta_0} \leq N\varepsilon^{-2-\delta_0+\delta},$$

where the constant $N = N(T, K, d, d_1)$.

Proof. Using Lemma 5 and the inequality $\varepsilon \geq h^{\delta_0}$, we have for $s \in [0, T]$ and $x \in R^d$

$$\begin{aligned}\Big|\frac{\partial}{\partial s}u_h^{(\varepsilon)}(s,x)\Big| &= \varepsilon^{-2}\Big|\int_{R^{d+1}} u_h(s-\varepsilon^2 r, x-\varepsilon\xi)\frac{\partial}{\partial s}\zeta(r,\xi)\,dr\,d\xi\Big| \\ &= \varepsilon^{-2}\Big|\int_{R^{d+1}} [u_h(s-\varepsilon^2 r, x-\varepsilon\xi) - u_h(s,x)]\frac{\partial}{\partial s}\zeta(r,\xi)\,dr\,d\xi\Big| \\ &\leq N\varepsilon^{-2}(h^{\delta\delta_0}+\varepsilon^\delta) \leq N\varepsilon^{-2+\delta} \qquad (3.4)\end{aligned}$$

and

$$\begin{aligned}\Big|\frac{\partial}{\partial x_i}u_h^{(\varepsilon)}(s,x)\Big| &= \varepsilon^{-1}\Big|\int_{R^{d+1}} u_h(s-\varepsilon^2 r, x-\varepsilon\xi)\frac{\partial}{\partial x_i}\zeta(r,\xi)\,dr\,d\xi\Big| \\ &= \varepsilon^{-1}\Big|\int_{R^{d+1}} [u_h(s-\varepsilon^2 r, x-\varepsilon\xi) - u_h(s,x)]\frac{\partial}{\partial x_i}\zeta(r,\xi)\,dr\,d\xi\Big| \\ &\leq N\varepsilon^{-1}(h^{\delta\delta_0}+\varepsilon^\delta) \leq N\varepsilon^{-1+\delta}. \qquad (3.5)\end{aligned}$$

Similarly,

$$\Big|\frac{\partial^2}{\partial x_i\partial x_j}u_h^{(\varepsilon)}\Big|_0 \leq N\varepsilon^{-2+\delta}, \qquad (3.6)$$

$$\Big|\frac{\partial^2}{\partial s\partial x_i}u_h^{(\varepsilon)}\Big|_0 + \Big|\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}u_h^{(\varepsilon)}\Big|_0 \leq N\varepsilon^{-3+\delta}, \qquad (3.7)$$

$$\Big|\frac{\partial^2}{\partial s^2}u_h^{(\varepsilon)}\Big|_0 + \Big|\frac{\partial^3}{\partial s\partial x_i\partial x_j}u_h^{(\varepsilon)}\Big|_0 \leq N\varepsilon^{-4+\delta}. \qquad (3.8)$$

Let us prove the inequality

$$\Big[\frac{\partial}{\partial s}u_h^{(\varepsilon)}\Big]_{\frac{\delta_0}{2},\delta_0} \leq N\varepsilon^{-2-\delta_0+\delta}. \qquad (3.9)$$

Assume that $|t-s|^{\frac{1}{2}} + |x-y| \leq \varepsilon$. Then using (3.7) and (3.8) we have

$$\begin{aligned}\Big|\frac{\partial}{\partial s}u_h^{(\varepsilon)}(s,x) - \frac{\partial}{\partial s}u_h^{(\varepsilon)}(t,y)\Big| &\leq |t-s|\Big|\frac{\partial^2}{\partial s^2}u_h^{(\varepsilon)}\Big|_0 + N|x-y|\Big|\frac{\partial^2}{\partial s\partial x_i}u_h^{(\varepsilon)}\Big|_0 \\ &\leq N(|t-s|\varepsilon^{-4+\delta} + |x-y|\varepsilon^{-3+\delta}) \\ &\leq N\varepsilon^{-2-\delta_0+\delta}(|t-s|^{\frac{\delta_0}{2}} + |x-y|^{\delta_0}).\end{aligned}$$

Assume that $|t-s|^{\frac{1}{2}} + |x-y| \geq \varepsilon$. Then using (3.4) we have

$$\begin{aligned}\Big|\frac{\partial}{\partial s}u_h^{(\varepsilon)}(s,x) - \frac{\partial}{\partial s}u_h^{(\varepsilon)}(t,y)\Big| &\leq 2\Big|\frac{\partial}{\partial s}u_h^{(\varepsilon)}\Big|_0 \leq N\varepsilon^{-2+\delta} \\ &\leq N\varepsilon^{-2-\delta_0+\delta}(|t-s|^{\frac{\delta_0}{2}} + |x-y|^{\delta_0}),\end{aligned}$$

and (3.9) is proved.

Using similar arguments, Assumption 1 and estimates (3.4)–(3.8), we easily prove that for each $\alpha \in A$, $i,j = 1,\dots,d$

$$\Big[\frac{\partial^2}{\partial x_i \partial x_j} u_h^{(\varepsilon)}\Big]_{\frac{\delta_0}{2},\delta_0} + \Big[\frac{\partial}{\partial x_i} u_h^{(\varepsilon)}\Big]_{\frac{\delta_0}{2},\delta_0} \le N\varepsilon^{-2-\delta_0+\delta}, \tag{3.10}$$

$$\Big[a_{ij}(\alpha,\cdot,\cdot)\frac{\partial^2}{\partial x_i \partial x_j} u_h^{(\varepsilon)}(\cdot,\cdot)\Big]_{\frac{\delta_0}{2},\delta_0} + \Big[b_i(\alpha,\cdot,\cdot)\frac{\partial}{\partial x_i} u_h^{(\varepsilon)}(\cdot,\cdot)\Big]_{\frac{\delta_0}{2},\delta_0} \le N\varepsilon^{-2-\delta_0+\delta}.$$

It remains to prove the estimate

$$\Big[\int \nabla^2_{c(\alpha,\cdot,\cdot,z)} u_h^{(\varepsilon)}(\cdot,\cdot)\Pi(dz)\Big]_{\frac{\delta_0}{2},\delta_0} \le N\varepsilon^{-2-\delta_0+\delta}.$$

Obviously,

$$\begin{aligned}
&\Big|\int \nabla^2_{c(\alpha,s,x,z)} u_h^{(\varepsilon)}(s,x)\Pi(dz) - \int \nabla^2_{c(\alpha,t,y,z)} u_h^{(\varepsilon)}(t,y)\Pi(dz)\Big| \\
&\quad \le \int |\nabla^2_{c(\alpha,s,x,z)} u_h^{(\varepsilon)}(s,x) - \nabla^2_{c(\alpha,s,x,z)} u_h^{(\varepsilon)}(t,y)|\Pi(dz) \\
&\qquad + \int |\nabla^2_{c(\alpha,s,x,z)} u_h^{(\varepsilon)}(t,y) - \nabla^2_{c(\alpha,t,y,z)} u_h^{(\varepsilon)}(t,y)|\Pi(dz) \equiv I_1 + I_2.
\end{aligned}$$

Using the formula

$$\nabla^2_y u_h^{(\varepsilon)}(s,x) = \sum_{i,j=1}^d \int_0^1 (1-\Theta)\frac{\partial^2 u_h^{(\varepsilon)}}{\partial x_i \partial x_j}(s, x+\Theta y)\, d\Theta y_i y_j,$$

(3.10) and Assumption 1, we have

$$\begin{aligned}
I_1 &\le \sum_{i,j=1}^d \int \Big|\int_0^1 (1-\Theta)\Big[\frac{\partial^2}{\partial x_i \partial x_j} u_h^{(\varepsilon)}(s, x+\Theta c(\alpha,s,x,z)) \\
&\quad - \frac{\partial^2}{\partial x_i \partial x_j} u_h^{(\varepsilon)}(t, y+\Theta c(\alpha,s,x,z))\Big] d\Theta c_i(\alpha,s,x,z) c_j(\alpha,s,x,z)\Big|\Pi(dz) \\
&\le N \max_{i,j=1,\dots,d}\Big[\frac{\partial^2}{\partial x_i \partial x_j} u_h^{(\varepsilon)}\Big]_{\frac{\delta_0}{2},\delta_0} \|c(\alpha,s,x,\cdot)\|^2_{2,\Pi} \\
&\quad \times (|t-s|^{\frac{\delta_0}{2}} + |x-y|^{\delta_0}) \le N\varepsilon^{-2-\delta_0+\delta}(|t-s|^{\frac{\delta_0}{2}} + |x-y|^{\delta_0}).
\end{aligned}$$

To estimate I_2, we notice that for each $\xi, \eta \in R^d$

$$\begin{aligned}
&|\nabla^2_\xi u_h^{(\varepsilon)}(t,y) - \nabla^2_\eta u_h^{(\varepsilon)}(t,y)| \\
&\quad = \Big|\sum_{i=1}^d \int_0^1 \big[\frac{\partial}{\partial x_i} u_h^{(\varepsilon)}(t, y+\eta+\Theta(\xi-\eta)) - \frac{\partial}{\partial x_i} u_h^{(\varepsilon)}(t,y)\big] d\Theta(\xi_i - \eta_i)\Big| \\
&\quad \le N \max_{i,j=1,\dots,d}\Big|\frac{\partial^2}{\partial x_i \partial x_j} u_h^{(\varepsilon)}\Big|_0 (|\xi| + |\eta|)|\xi - \eta|.
\end{aligned}$$

Therefore, by Hölder's inequality, (3.6) and Assumption 1,

$$
\begin{aligned}
I_2 &\le N\varepsilon^{-2+\delta}\int(|c(\alpha,s,x,z)|+|c(\alpha,t,y,z)|)|c(\alpha,s,x,z)-c(\alpha,t,y,z)|\Pi(dz)\\
&\le N\varepsilon^{-2+\delta}(\|c(\alpha,s,x,\cdot)\|_{2,\Pi}+\|c(\alpha,t,y,\cdot)\|_{2,\Pi})\|c(\alpha,s,x,\cdot)-c(\alpha,t,y,\cdot)\|_{2,\Pi}\\
&\le N\varepsilon^{-2+\delta}(|t-s|^{\frac{\delta_0}{2}}+|x-y|\wedge 1)\le N\varepsilon^{-2+\delta}(|t-s|^{\frac{\delta_0}{2}}+|x-y|^{\delta_0}.
\end{aligned}
$$

The lemma is proved. □

4 Proof of Main Results

Proof of Theorem 1. Since $v_h \le v$, it suffices to prove that for each $s\in[0,T]$ and $x\in R^d$

$$
v(s,x)\le v_h(s,x)+Nh^{\varkappa}. \tag{4.1}
$$

If $s\in[T-h^2,T]$, then, by Remark 3, for each $\alpha\in\mathfrak{A}$

$$
|v^\alpha(s,x)-g(x)|\le \mathbb{E}|g(X^{\alpha,s,x}_{T-s})-g(x)|\le K\mathbb{E}|X^{\alpha,s,x}_{T-s}-x|^\delta\le Nh^\delta.
$$

Thus we have to prove (4.1) for $s\le T-h^2$ assuming without loss of generality that $T\ge h^2$. Denote $\varepsilon=h^{\varkappa/(\delta\delta_0)}$. Let $\beta=(\alpha,r,\xi)\in\mathfrak{B}_h$ and $t\le h^2$. As can be easily seen,

$$
X^{\beta,s,x}_t = X^{\alpha,s+\varepsilon^2 r,x+\varepsilon\xi}_t-\varepsilon\xi.
$$

Therefore, by Remark 2, for each $\beta=(\alpha,r,\xi)\in B$, $s\in[0,S-h^2]$ and $x\in R^d$

$$
u_h(s,x)\ge \mathbb{E}u_h(s+h^2,X^{\beta,s,x}_{h^2})=\mathbb{E}u_h(s+h^2,X^{\alpha,s+\varepsilon^2 r,x+\varepsilon\xi}_{h^2}-\varepsilon\xi)
$$

or for each $\alpha\in A$, $r\in(-1,0)$, $\xi\in B_1$, $s\le S-h^2+\varepsilon^2 r$ and $x\in R^d$

$$
u_h(s-\varepsilon^2 r,x-\varepsilon\xi)\ge \mathbb{E}u_h(s-\varepsilon^2 r+h^2,X^{\alpha,s,x}_{h^2}-\varepsilon\xi).
$$

Multiplying the last inequality by the smooth kernel ζ defined in Section 3 and integrating, we get for each $\alpha\in A$, $s\le T-h^2$ and $x\in R^d$

$$
u^{(\varepsilon)}_h(s,x)\ge \mathbb{E}u^{(\varepsilon)}_h(s+h^2,X^{\alpha,s,x}_{h^2}).
$$

This inequality, together with Itô's formula, implies

$$
\mathbb{E}\int_0^{h^2}L^\alpha u^{(\varepsilon)}_h(s+r,X^{\alpha,s,x}_r)\,dr\le 0.
$$

Hence, by Remark 3 and Lemma 6, for each $\alpha\in A$, $s\le T-h^2$ and $x\in R^d$

$$
\begin{aligned}
L^\alpha u^{(\varepsilon)}_h(s,x) &\le \frac{1}{h^2}\mathbb{E}\int_0^{h^2}[L^\alpha u^{(\varepsilon)}_h(s,x)-L^\alpha u^{(\varepsilon)}_h(s+r,X^{\alpha,s,x}_r)]\,dr\\
&\le [L^\alpha u^{(\varepsilon)}_h]_{\frac{\delta_0}{2},\delta_0}\frac{1}{h^2}\int_0^{h^2}(r^{\frac{\delta_0}{2}}+\mathbb{E}|X^{\alpha,s,x}_r-x|^{\delta_0})\,dr\le N\varepsilon^{-2-\delta_0+\delta}h^{\delta_0}.
\end{aligned}
$$

Therefore, by Itô's formula, for each $\alpha \in \mathfrak{A}$, $s \le T - h^2$ and $x \in R^d$

$$\mathbb{E}u_h^{(\varepsilon)}(T-h^2, X^{\alpha,s,x}_{T-h^2-s}) - u_h^{(\varepsilon)}(s,x)$$
$$= \mathbb{E}\int\limits_0^{T-h^2-s} L^{\alpha_t} u_h^{(\varepsilon)}(s+t, X_t^{\alpha,s,x})\,dt \le N\varepsilon^{-2-\delta_0+\delta}h^{\delta_0}.$$

Since, by Lemma 5,

$$|u_h^{(\varepsilon)}(s,x) - u_h(s,x)| \le \int\limits_{R^{d+1}} |u_h(s-\varepsilon^2 r, x-\varepsilon\xi) - u_h(s,x)|\zeta(r,\xi)\,dr\,d\xi$$
$$\le N(h^{\delta\delta_0} + \varepsilon^{\delta}) \le N\varepsilon^{\delta},$$

we have

$$\mathbb{E}u_h(T-h^2, X^{\alpha,s,x}_{T-h^2-s}) \le u_h(s,x) + N(\varepsilon^{\delta} + \varepsilon^{-2-\delta_0+\delta}h^{\delta_0}).$$

Furthermore, by Lemma 5,

$$|u_h(T-h^2,x) - g(x)| = |u_h(T-h^2,x) - u_h(S,x)|$$
$$\le N((\varepsilon^2+h^2)^{\frac{\delta}{2}} + h^{\delta\delta_0}) \le N\varepsilon^{\delta}.$$

Hence

$$\mathbb{E}g(X^{\alpha,s,x}_{T-h^2-s}) \le u_h(s,x) + N(\varepsilon^{\delta} + \varepsilon^{-2-\delta_0+\delta}h^{\delta_0}).$$

and, by Remark 3 and Lemma 4,

$$\begin{aligned} v^{\alpha}(s,x) &= \mathbb{E}g(X^{\alpha,s,x}_{T-s}) \le \mathbb{E}|g(X^{\alpha,s,x}_{T-s}) - g(X^{\alpha,s,x}_{T-s-h^2})| + \mathbb{E}g(X^{\alpha,s,x}_{T-s-h^2}) \\ &\le K\mathbb{E}|X^{\alpha,s,x}_{T-s} - X^{\alpha,s,x}_{T-s-h^2}|^{\delta} + \mathbb{E}g(X^{\alpha,s,x}_{T-s-h^2}) \\ &\le u_h(s,x) + N(h^{\delta} + \varepsilon^{\delta} + \varepsilon^{-2-\delta_0+\delta}h^{\delta_0}) \\ &\le v_h(s,x) + N(\varepsilon^{\delta_0\delta}\varepsilon^{-2-\delta_0+\delta}h^{\delta_0}). \end{aligned}$$

The theorem is proved. □

Proof of Theorem 2. Fix $h \in (0,1)$ and for $\alpha \in \mathfrak{B}_h$ define the process $\overline{x}_t = \overline{x}_t^{\alpha,s,x}(h)$ recursively by

$$\begin{aligned} \overline{x}_0 &= x, \\ \overline{x}_t &= \overline{x}_{nh^2} + \sigma(\alpha_{nh^2}, s+nh^2, \overline{x}_{nh^2})(W_t - W_{nh^2}) \\ &\quad + b(\alpha_{nh^2}, s+nh^2, \overline{x}_{nh^2})(t-nh^2) \\ &\quad + \int_{nh^2}^t \int_{R^m} c(\alpha_{nh^2}, s+nh^2, \overline{x}_{nh^2}, z)\,q(dr,dz) \end{aligned}$$

for $nh^2 \le t \le (n+1)h^2$. It is easy to see that $\overline{x}_t^{\alpha,s,x}(h)$ satisfies the equation

$$\begin{aligned} \overline{x}_t = x &+ \int_0^t \sigma(\alpha_r, s+\varkappa_h(r), \overline{x}_{\varkappa_h(r)})\,dW_r + \int_0^t b(\alpha_r, s+\varkappa_h(r), \overline{x}_{\varkappa_h(r)})\,dr \\ &+ \int_0^t \int_{R^m} c(\alpha_r, s+\varkappa_h(r), \overline{x}_{\varkappa_h(r)}, z)\,q(dr,dz), \end{aligned} \tag{4.2}$$

where $\varkappa_h(t) = h^2[\frac{t}{h^2}]$. Let

$$\overline{v}_h^\alpha(s,x) = \mathbb{E} g(\overline{x}^{\alpha,s,x}_{\varkappa_h(T-s)}(h)).$$

Notice that the function $\overline{v}_h$ defined in Section 2 satisfies the dynamic programming equation for the problem of maximizing $\overline{v}_h^\alpha(s,x)$ over $\alpha \in \mathfrak{A}_h$. Therefore,

$$\overline{v}_h(s,x) = \sup_{\alpha\in\mathfrak{A}_h} \overline{v}_h^\alpha(s,x).$$

Rewrite (4.2) as

$$\overline{x}_t = x + \int_0^t \overline{\sigma}_r(\overline{x}_r)\,dW_r + \int_0^t \overline{b}_r(\overline{x}_r)\,dr + \int_0^t\int_{R^m} \overline{c}_r(\overline{x}_r,z)\,q(dr,dz),$$

where

$$\overline{\sigma}_t(y) = \sigma(\alpha_t, s+\varkappa_h(t), y + \overline{x}^{\alpha,s,x}_{\varkappa_h(t)}(h) - \overline{x}^{\alpha,s,x}_t(h))$$

and similarly are defined $\overline{b}_t(y)$ and $\overline{c}_t(y,z)$. Then, by Lemma 2, for each $\alpha \in \mathfrak{A}_h$, $s \in [0,T]$ and $x \in R^d$

$$\begin{aligned}
\mathbb{E}\sup_{t\le T} |\overline{x}^{\alpha,s,x}_t - X^{\alpha,s,x}_t|^2 &\le N\mathbb{E}\int_0^T \big(\|\sigma(\alpha_t, s+t, X^{\alpha,s,x}_t) - \overline{\sigma}_t(X^{\alpha,s,x}_t)\|^2 \\
&\quad + |b(\alpha_t, s+t, X^{\alpha,s,x}_t) - \overline{b}_t(X^{\alpha,s,x}_t)|^2 \\
&\quad + \|c(\alpha_t, s+t, X^{\alpha,s,x}_t,\cdot) - \overline{c}_t(X^{\alpha,s,x}_t,\cdot)\|^2_{2,\Pi}\big)\,dt \\
&\le N\int_0^t \big(h^{2\delta_0} + \mathbb{E}|\overline{x}^{\alpha,s,x}_{\varkappa_h(t)}(h) - \overline{x}^{\alpha,s,x}_t(h)|^2\big)\,dt \le Nh^{2\delta_0}.
\end{aligned}$$

This estimate and Remark 3 imply that for each $\alpha \in \mathfrak{A}_h$, $s \in [0,T]$ and $x \in R^d$

$$\begin{aligned}
|v_h^\alpha(s,x) - \overline{v}_h^\alpha(s,x)| &\le \mathbb{E}|g(X^{\alpha,s,x}_{(T-s)}) - g(\overline{x}^{\alpha,s,x}_{\varkappa_h(T-s)}(h))| \\
&\le K\mathbb{E}|X^{\alpha,s,x}_{(T-s)} - \overline{x}^{\alpha,s,x}_{\varkappa_h(T-s)}(h)|^\delta \le Nh^{\delta\delta_0}.
\end{aligned}$$

Hence, for each $s \in [0,T]$ and $x \in R^d$

$$|v_h(s,x) - \overline{v}_h(s,x)| \le Nh^{\delta\delta_0}$$

what, together with Theorem 1, implies the assertion of the theorem. □

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