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On a Two-Fluid Inclined Film Flow with Evaporation

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Abstract. This paper is concerned with a plane steady-state inclined film flow including evaporation effects. The motion is governed by a free boundary value problem for a coupled system of Navier–Stokes and Stefan equations. The flow domain is unbounded in two directions and it contains a geometrical perturbation on the inclined bottom. Existence and uniqueness of a suitable solution in weighted Sobolev spaces can be proved for small data (perturbation, inclination of the bottom) characterizing the problem.

Keywords: Navier–Stokes equations, Stefan equations, evaporation, free boundary value problem, perturbed bottom.

AMS Subject Classification: 35R35; 35Q30; 76D03; 76D05.

1 Introduction

The paper is devoted to a plane stationary nonisothermal two-fluid flow problem with two free boundaries describing an inclined film flow with evaporation. Two heavy viscous incompressible and heat-conducting fluids are flowing down a geometrically perturbed inclined bottom (cf. Figure 1). Both of the a priori unknown free boundaries are noncompact in two directions. The flow domain is unbounded and evaporation is taken into account. This is quite important in several technological and scientific applications; interesting examples may be found in the field of materials science, particularly in coating and solidification processes with evaporation (cf. [2, 4, 5, 8, 10, 17, 21, 25]) or in crystal-growth processes [13, 16, 18].

In this paper we investigate a problem for a 2D steady-state flow with two viscous incompressible heat-conducting fluids (having kinematic viscosities $\nu_i > 0$, densities $\varrho_i > 0$ and thermal conductivities λ_i, $i = 1, 2$) down an inclined bottom S_0 having a slope α (cf. Figure 1). In fact, the bottom S_0 represents a perturbed plane. Suppose that the bottom is given by the formula $S_0 = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \colon x_2 = \varepsilon\varphi_0(x_1),\ -\infty < x_1 < +\infty\}$ with φ_0 having a compact support, i.e. $\varphi_0(x_1) \equiv 0$ for $|x_1| \geqslant R_0 > 0$, and assume that the direction $\boldsymbol{e}_g$

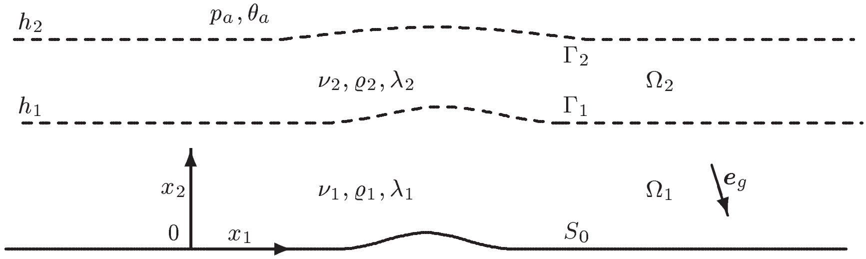

Figure 1. Flow domain of a nonisothermal two-fluid inclined film flow.

of the gravity is the vector $\boldsymbol{e}_g = (\sin\alpha, -\cos\alpha)^T$ which makes (with respect to the given co-ordinate system) an angle $\alpha^* := \pi/2 - \alpha$ $(0 < \alpha \leqslant \pi/2)$ with the x_1-axis. Note that the corresponding problem will be formulated in dimensionless form. The concrete transition to that formulation can be found in [20].

Now let us explain the problem in more detail. We are concerned with the two-fluid flow down the inclined bottom S_0 caused by gravity $g\,\boldsymbol{e}_g$, only. This means mathematically that the positive layer thickness at infinity in each liquid layer Ω_i $(i = 1, 2)$ is *a priori* given. For example, in slide coaters such kind of motions occur on some parts of the coater. The flow fields and layer profiles are essential for associated problems.

Assume that the free interface Γ_1 separating the two fluids and the upper free surface Γ_2 admit the parametrizations $\Gamma_i = \{\boldsymbol{x} \in \mathbb{R}^2 \colon x_2 = \psi_i(x_1),\ -\infty < x_1 < +\infty\}$ $(i = 1, 2)$, where the functions ψ_i $(i = 1, 2)$ are a priori unknown and have to be determined. Let $h_i > 0$ $(0 < h_1 < h_2)$ be the prescribed constant limits of $\psi_i(x_1)$ $(i = 1, 2)$, at infinity. The problem under study has the following form: to find a vector of velocity $\boldsymbol{v} = (v_1(x_1, x_2), v_2(x_1, x_2))^T$, a pressure $p(x_1, x_2)$, a temperature $\theta(x_1, x_2)$ and functions $\psi_i(x_1)$ $(i = 1, 2)$ satisfying in the domain $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \{\boldsymbol{x} \in \mathbb{R}^2 \colon \varepsilon\varphi_0(x_1) < x_2 < \psi_1(x_1),\ -\infty < x_1 < +\infty\}$ and $\Omega_2 = \{\boldsymbol{x} \in \mathbb{R}^2 \colon \psi_1(x_1) < x_2 < \psi_2(x_1),\ -\infty < x_1 < +\infty\}$ the following system of equations for a coupled heat- and mass transfer process

$$\begin{cases} (\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{v} - \nu\,\boldsymbol{\nabla}^2\boldsymbol{v} + \dfrac{1}{\varrho}\,\boldsymbol{\nabla} p = g\,\boldsymbol{e}_g, \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \\ (\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\theta - \lambda\,\boldsymbol{\nabla}^2\theta = 0, \end{cases} \tag{1.1}$$

and the boundary conditions on S_0

$$\boldsymbol{v}|_{S_0} = \boldsymbol{0}, \quad \theta|_{S_0} = \theta_0. \tag{1.2}$$

The boundary conditions on the free interface Γ_1 are the following ones

$$\begin{cases} [\theta]|_{\Gamma_1} = 0, \qquad [\boldsymbol{v}]|_{\Gamma_1} = \boldsymbol{0}, \\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma_1^-} = \left[\lambda\,\dfrac{\partial\theta}{\partial n}\right]\Big|_{\Gamma_1}, \quad \big[\boldsymbol{\tau}\cdot\boldsymbol{S}(\boldsymbol{v})\,\boldsymbol{n}\big]\big|_{\Gamma_1} = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}x_1}\dfrac{\psi_1'(x_1)}{\sqrt{1+\psi_1'(x_1)^2}} = \dfrac{1}{\sigma_1(\theta)}\big[-p+\boldsymbol{n}\cdot\boldsymbol{S}(\boldsymbol{v})\,\boldsymbol{n}\big]\big|_{\Gamma_1}, \\ \lim\limits_{|x_1|\to+\infty}\psi_1(x_1) = h_1. \end{cases} \tag{1.3}$$

Finally, at the free surface Γ_2 we are given the boundary conditions

$$\begin{cases} \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma_2} = \left(\theta-\theta_a+\lambda_2\,\dfrac{\partial\theta}{\partial n}\right)\Big|_{\Gamma_2}, \quad \boldsymbol{\tau}\cdot\boldsymbol{S}(\boldsymbol{v})\,\boldsymbol{n}|_{\Gamma_2} = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}x_1}\dfrac{\psi_2'(x_1)}{\sqrt{1+\psi_2'(x_1)^2}} = \dfrac{1}{\sigma_2(\theta)}\big(p_a-p+\boldsymbol{n}\cdot\boldsymbol{S}(\boldsymbol{v})\,\boldsymbol{n}\big)\big|_{\Gamma_2}, \\ \lim\limits_{|x_1|\to+\infty}\psi_2(x_1) = h_2. \end{cases} \tag{1.4}$$

It is well-known (cf. [3]) that for a large number of fluids the surface tensions σ_i can be represented as linear functions of the temperature θ along the free interface Γ_i $(i=1,2)$ (cf. also [13, 19]) as follows

$$\sigma_i(\theta) = a_i - b_i\theta \quad (a_i, b_i > 0;\ i=1,2). \tag{1.5}$$

Under λ_m we understand the thermal conductivity of the m-th fluid $(m=1,2)$ in Problem (1.1)–(1.4). The symbol g means the acceleration of gravity. The value θ_0 denotes the (constant) given temperature of the wall S_0. Without loss of generality one can assume that $\theta_0 = 0$ and that θ is in fact the difference between the physical temperature and θ_0. With p_a and θ_a we mention the given (constant) pressure and temperature of the ambient air, respectively.

Further, the following notations have been used: $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are unit vectors normal and tangential to Γ_1 and oriented as x_2, x_1, respectively. By $\boldsymbol{a}\cdot\boldsymbol{b}$ we denote the inner product of $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^2$, $\boldsymbol{\nabla} = (\partial/\partial x_1, \partial/\partial x_2)^T$ is the gradient operator, $\boldsymbol{\nabla}p = \operatorname{grad}p$, $\boldsymbol{\nabla}\cdot\boldsymbol{v} = \operatorname{div}\boldsymbol{v}$, $\varrho|_{\Omega_m} = \varrho_m$ $(m=1,2)$ declares the restriction of ϱ to Ω_m (analogously for ν and λ). $\boldsymbol{\nabla}^2$ denotes the Laplace operator. By $\boldsymbol{S}(\boldsymbol{v})$ we denote the deviatoric stress tensor, i.e. a matrix with elements $S_{ij}(\boldsymbol{v}) = 0.5\varrho\nu(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ $(i,j=1,2)$. The symbol $[w]|_{\Gamma_1}$ is called the jump of w crossing the free interface Γ_1, that is,

$$\big[w(\boldsymbol{x}_0)\big]\big|_{\Gamma_1} := \lim_{\boldsymbol{y}\to\boldsymbol{x}_0} w(\boldsymbol{y}) - \lim_{\boldsymbol{x}\to\boldsymbol{x}_0} w(\boldsymbol{x}) \quad (\boldsymbol{x}_0\in\Gamma_1,\ \boldsymbol{y}\in\Omega_1,\ \boldsymbol{x}\in\Omega_2), \tag{1.6}$$

and the symbol $w|_{\Gamma_1^-}$ declares the limit from below at the interface Γ_1; more precisely

$$w(\boldsymbol{x}_0)|_{\Gamma_1^-} := \lim_{\boldsymbol{y}\to\boldsymbol{x}_0} w(\boldsymbol{y}) \quad (\boldsymbol{x}_0\in\Gamma_1,\ \boldsymbol{y}\in\Omega_1). \tag{1.7}$$

Note that the left-hand side of $(1.3)_5$ (i.e. of the fifth equation in (1.3)) is equal to the curvature $K_1(x_1)$ of Γ_1. The same is true for $K_2(x_1)$ in case of Γ_2. Furthermore, Equations $(1.3)_3$ and $(1.4)_1$ represent the mathematical expressions of evaporation effects.

2 General Solution Scheme

Free boundary value problems for the stationary Navier–Stokes equations or their modifications were the topic of many mathematicians. Numerous references in this field can be found, e.g., in the bibliographies of [9, 12, 23, 24]. In the mathematical papers [6, 7, 10, 13, 15, 19, 21, 22] the dependence on temperature was additionally taken into account. Some of them are devoted to problems with thermocapillary convection whereas others include the effect of evaporation. In [25] a one-dimensional approximation of an evaporation problem was analytically studied. Computational investigations of nonisothermal free boundary problems are described in the studies [3, 14, 20] and many others.

For free boundary value problems where the unknown flow domain is unbounded in two directions as in Problem (1.1)–(1.4) a special linearization procedure was necessary (cf. [9, 12] and others).

Solving such kind of problems – in [9], and independently in [1], an appropriate procedure was proposed which is based on a linearization of the original problem on a corresponding exact solution in the unperturbed "uniform" flow domain, say $\mathit{\Pi} = \{\boldsymbol{x} \in \mathbb{R}^2 \colon 0 < x_2 < h_1 \vee h_1 < x_2 < h_2\}$. The difference of this technique from previous applied ones is that on each step of iterations the determination of $\boldsymbol{v}$, p, θ is not separated from the determination of the free boundaries $\mathit{\Gamma}_i$ $(i = 1, 2)$ (i.e. from the determination of the functions ψ_i describing $\mathit{\Gamma}_i$). For Problem (1.1)–(1.4) this scheme can be characterized by the diagram

$$\big(\boldsymbol{v}^0, p^0, \theta^0, \psi_1^0, \psi_2^0\big) \to \big(\boldsymbol{v}^1, p^1, \theta^1, \psi_1^1, \psi_2^1\big) \to \cdots \to \big(\boldsymbol{v}^m, p^m, \theta^m, \psi_1^m, \psi_2^m\big) \to \cdots$$

where on each step of iterations the linearized problem is solved in the same "strip-like" domain and the functions $\boldsymbol{v}$, p, θ and ψ_i $(i = 1, 2)$ are determined simultaneously. Note that the superscripts in the above formula denote the number of the corresponding iteration step.

An important part in the derivation of the correct linearization takes the calculation of exact solutions of the nonlinear problems in a "uniform" (not perturbed) flow domain. These exact (basic) solutions in the uniform domain $\mathit{\Pi}$ will be determined in Section 4 – Basic solutions. They are also useful for the numerical flow simulation: they can be used as inlet boundary data in more complicated problems. In [12] the analogous isothermal problem (without any inclusion of temperature) to Problem (1.1)–(1.4) was solved by numerical methods. Finally, in [22] a problem was studied which has a lot of common features with the presented one. Instead of evaporation thermocapillary convection of Marangoni type was considered there.

3 Function Spaces

When studying Problem (1.1)–(1.4) it is useful to work with weighted Sobolev spaces. Let $\mathit{\Pi}_m$ $(m = 1, 2)$ be the strip-like domains

$$\mathit{\Pi}_1 := \big\{\boldsymbol{x} \in \mathbb{R}^2 \colon 0 < x_2 < h_1,\ -\infty < x_1 < +\infty\big\},$$
$$\mathit{\Pi}_2 := \big\{\boldsymbol{x} \in \mathbb{R}^2 \colon h_1 < x_2 < h_2,\ -\infty < x_1 < +\infty\big\},$$

and $\Pi = \Pi_1 \cup \Pi_2$ their union. We introduce the space $W^{l,2}_\beta(\Pi)$ of functions $\boldsymbol{u}$ on Π with restrictions $\boldsymbol{u}^{(m)} = \boldsymbol{u}|_{\Pi_m}$ belonging to $W^{l,2}_\beta(\Pi_m)$ $(m = 1, 2)$ and having the finite norms

$$\big\|\boldsymbol{u}^{(m)}; W^{l,2}_\beta(\Pi_m)\big\| = \big\|\boldsymbol{u}^{(m)} \exp\big(\beta\sqrt{1 + x_1^2}\big); W^{l,2}(\Pi_m)\big\| \quad (m = 1, 2),$$

where $W^{l,2}(\Pi_m)$ is the usual Sobolev space. The norm in $W^{l,2}_\beta(\Pi)$ is given by

$$\big\|\boldsymbol{u}; W^{l,2}_\beta(\Pi)\big\| = \sum_{m=1}^{2} \big\|\boldsymbol{u}^{(m)} \exp\big(\beta\sqrt{1 + x_1^2}\,\big); W^{l,2}(\Pi_m)\big\|.$$

If $\beta > 0$, then elements of $W^{l,2}_\beta(\Pi)$ vanish exponentially as $|x_1| \to \infty$ and, if $\beta < 0$, then elements $\boldsymbol{u} \in W^{l,2}_\beta(\Pi)$ might exponentially increase as $|x_1| \to \infty$.

The spaces $W^{l-1/2,2}_\beta(\mathbb{R})$ of functions defined on $\mathbb{R}$ can be introduced analogously. Let $S = \{x \in \Pi\colon x_1 \in \mathbb{R},\ x_2 = h \in \{0, h_1, h_2\}\}$ be a line. Denote by $W^{l-1/2,2}_\beta(S)$ the spaces of traces on S of functions from $W^{l,2}_\beta(\Pi)$. Then $W^{l-1/2,2}_\beta(\mathbb{R})$ coincides with $W^{l-1/2,2}_\beta(S)$, i.e. if $\boldsymbol{u} \in W^{l-1/2,2}_\beta(\Pi)$, then $\boldsymbol{u}(\cdot, h) \in W^{l-1/2,2}_\beta(\mathbb{R})$).

In the paper the spaces of scalar and vector-valued functions are not distinguished in notations. The norm for vector-valued functions is then the sum of the norms of the corresponding coordinate functions.

4 Basic Flows

Firstly, let us introduce the flow parameter $r := \nu_1\varrho_1/(\nu_2\varrho_2)$ denoting the ratio of dynamic viscosities. In order to obtain one exact solution to Problem (1.1)–(1.4) in the uniform (not perturbed) domain Π which is in fact an unbounded double-strip (cf. Section 3) we make the subsequent assumptions

$$v_2 \equiv 0, \qquad \frac{\partial p}{\partial x_1} \equiv 0, \qquad \frac{\partial \theta}{\partial x_1} \equiv 0.$$

The second assumption reflects the circumstance that the fluid flow is caused by gravity, only. The first equation together with the continuity condition imply $\partial v_1/\partial x_1 = 0$. Under these assumptions one gets the following reduced nonisothermal Navier–Stokes system in Π

$$-\nu\varrho\,\boldsymbol{\nabla}^2 v_1 = \varrho\, g \sin\alpha, \qquad \frac{\partial p}{\partial x_2} = -\varrho\, g\cos\alpha, \qquad -\lambda\,\boldsymbol{\nabla}^2\,\theta = 0, \tag{4.1}$$

equipped with the corresponding boundary conditions. The three equations from (4.1) are independent of each other. Finally, we obtain three independent boundary value problems for some ordinary differential equations. The first problem contains the unknown velocity component $v_1(x_2)$ and takes the form

$$\begin{cases} \nu_1 \varrho_1 \dfrac{\mathrm{d}^2 v_1^{(1)}}{\mathrm{d}x_2^2} = -\varrho_1 \, g \sin\alpha, \qquad \nu_2 \varrho_2 \dfrac{\mathrm{d}^2 v_1^{(2)}}{\mathrm{d}x_2^2} = -\varrho_2 \, g \sin\alpha, \\ v_1^{(1)}(0) = 0, \qquad \left.\dfrac{\mathrm{d}v_1^{(2)}}{\mathrm{d}x_2}\right|_{x_2=h_2} = 0, \\ \left.v_1^{(1)}\right|_{x_2=h_1} = \left.v_1^{(2)}\right|_{x_2=h_1}, \qquad \nu_1 \varrho_1 \left.\dfrac{\mathrm{d}v_1^{(1)}}{\mathrm{d}x_2}\right|_{x_2=h_1} = \nu_2 \varrho_2 \left.\dfrac{\mathrm{d}v_1^{(2)}}{\mathrm{d}x_2}\right|_{x_2=h_1}. \end{cases} \tag{4.2}$$

The next boundary value problem relates to the unknown pressure field $p(x_2)$:

$$\begin{cases} \dfrac{\mathrm{d}p^{(1)}}{\mathrm{d}x_2} = -\varrho_1 \, g \cos\alpha, \qquad \dfrac{\mathrm{d}p^{(2)}}{\mathrm{d}x_2} = -\varrho_2 \, g \cos\alpha, \\ \left.p^{(1)}\right|_{x_2=h_1} = \left.p^{(2)}\right|_{x_2=h_1}, \qquad \left.p^{(2)}\right|_{x_2=h_2} = p_a, \end{cases} \tag{4.3}$$

The last problem was established for the unknown temperature field $\theta(x_2)$:

$$\begin{cases} \dfrac{\mathrm{d}^2\theta^{(1)}}{\mathrm{d}x_2^2} = 0, \qquad \dfrac{\mathrm{d}^2\theta^{(2)}}{\mathrm{d}x_2^2} = 0, \\ \left.\theta^{(1)}\right|_{x_2=0} = \theta_0, \qquad \left.\left(\theta^{(2)} - \theta_a + \lambda_2 \dfrac{\mathrm{d}\theta^{(2)}}{\mathrm{d}x_2}\right)\right|_{x_2=h_2} = 0, \\ \left.\theta^{(1)}\right|_{x_2=h_1} = \left.\theta^{(2)}\right|_{x_2=h_1}, \qquad \lambda_1 \left.\dfrac{\mathrm{d}\theta^{(1)}}{\mathrm{d}x_2}\right|_{x_2=h_1} = \lambda_2 \left.\dfrac{\mathrm{d}\theta^{(2)}}{\mathrm{d}x_2}\right|_{x_2=h_1}, \end{cases} \tag{4.4}$$

The solution to the systems (4.2)–(4.4) sometimes in the literature is called Nusselt solution of inclined film flow. It can be calculated in a straightforward manner. The solution for the velocity takes the form

$$v_1^0(x_2) = \begin{cases} g \sin\alpha \left[-\dfrac{1}{2\nu_1} x_2^2 + \left(\dfrac{h_2 - h_1}{r\nu_2} + \dfrac{h_1}{\nu_1} \right) x_2 \right], & 0 \leqslant x_2 \leqslant h_1, \\ g \sin\alpha \left[-\dfrac{1}{2\nu_2} (h_2 - x_2)^2 + \dfrac{1}{2\nu_2} (h_2 - h_1)^2 \right. & \\ \qquad \left. + \dfrac{1}{2\nu_1} h_1^2 + \dfrac{1}{r\nu_2} (h_2 - h_1) h_1 \right], & h_1 \leqslant x_2 \leqslant h_2. \end{cases} \tag{4.5}$$

The determined pressure is described by

$$p^0(x_2) = \begin{cases} \left[(h_1 - x_2)\varrho_1 + (h_2 - h_1)\varrho_2 \right] g \cos\alpha + p_a, & 0 \leqslant x_2 \leqslant h_1, \\ (h_2 - x_2)\varrho_2 g \cos\alpha + p_a, & h_1 \leqslant x_2 \leqslant h_2. \end{cases} \tag{4.6}$$

Finally, the calculated temperature can be written as

$$\theta^0(x_2) = \begin{cases} \dfrac{\theta_a \lambda_2}{\lambda_2 h_1 + \lambda_1 (h_2 - h_1) + \lambda_1 \lambda_2} \, x_2, & 0 \leqslant x_2 \leqslant h_1, \\ \dfrac{\theta_a [\lambda_1 x_2 + h_1 (\lambda_2 - \lambda_1)]}{\lambda_2 h_1 + \lambda_1 (h_2 - h_1) + \lambda_1 \lambda_2}, & h_1 \leqslant x_2 \leqslant h_2. \end{cases} \tag{4.7}$$

The formula (4.7) was obtained under the assumption $\theta_0 = 0$ which can be made without loss of generality. The above mentioned solution implies the subsequent fluxes (i.e. flow rates)

$$F_1 = g\sin\alpha\left[\frac{1}{3\nu_1}h_1^3 + \frac{1}{2r\nu_2}h_1^2(h_2-h_1)\right],$$
$$F_2 = g\sin\alpha\left[\frac{1}{3\nu_2}(h_2-h_1)^3 + (h_2-h_1)\left(\frac{1}{2\nu_1}h_1^2 + \frac{1}{r\nu_2}h_1(h_2-h_1)\right)\right],$$

which are small if the inclination angle α is small.

5 Results

Theorem 1. *Let* $S_0 = \{\boldsymbol{x} \in \mathbb{R}^2 \colon x_2 = \varepsilon\varphi_0(x_1),\ -\infty < x_1 < +\infty\}$, $\varphi_0 \in W_\beta^{l+5/2,2}(\mathbb{R})$ *with* $l \geqslant 0$, $\beta = |\beta_0|\sin\alpha > 0$, *where* β_0 *is independent of* α *and* α *denotes the slope of the inclined bottom* S_0. *Assume that* α *is sufficiently small. Then there exist real numbers* $\widehat{\varepsilon}, \widehat{r} > 0$ *such that for arbitrary* $\varepsilon \in (0, \widehat{\varepsilon})$ *the Problem* (1.1)–(1.4) *has a unique solution* $(\boldsymbol{v}, p, \theta, \psi_1, \psi_2)^T$ *characterized by the structure*

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}^0(\boldsymbol{x}) + \varepsilon\,\boldsymbol{u}(\boldsymbol{x}), \quad p(\boldsymbol{x}) = p^0(\boldsymbol{x}) + \varepsilon\, q(\boldsymbol{x}), \quad \theta(\boldsymbol{x}) = \theta^0(\boldsymbol{x}) + \varepsilon\,\vartheta(\boldsymbol{x}),$$
$$\psi_1(x_1) = h_1 + \varepsilon\,\Psi_1(x_1), \quad \psi_2(x_1) = h_2 + \varepsilon\,\Psi_2(x_1),$$

where $\{\boldsymbol{v}^0, p^0, \theta^0\}$ *denotes the basic solution* (4.5)–(4.7) *and the regular part* $\boldsymbol{U}$ *has the representation*

$$\boldsymbol{U} := (\boldsymbol{u}, q, \vartheta, \Psi_1, \Psi_2)^T \in \left[W_\beta^{l+2,2}(\Pi)\right]^2 \times W_\beta^{l+1,2}(\Pi) \times W_\beta^{l+2,2}(\Pi)$$
$$\times \left[W_\beta^{l+5/2,2}(\mathbb{R})\right]^2 \equiv \mathfrak{D}_\beta^{l,2}W(\Pi).$$

Furthermore, the following inequalities hold

$$\left\|\boldsymbol{U}; \mathfrak{D}_\beta^{l,2}W(\Pi)\right\| \leqslant \widehat{r}, \qquad \widehat{\varepsilon} \leqslant \mathrm{const}\cdot\sin^2\alpha.$$

Let us shortly describe the main steps and ideas of the proof. We realize the proof by successive approximations. In a first step the original (perturbed) and unknown flow domain Ω (cf. Figure 1) is transformed onto the uniform (strip-like) domain Π. Then, by using the transformation mapping, the original flow Problem (1.1)–(1.4) is linearized over the basic solution (4.5)–(4.7) in domain Π. By $\mathfrak{N}$ we assign the operator on the left-hand side of the corresponding linearized auxiliary problem. Thanks to a related theorem for the linear auxiliary problem (see, e.g., [12]), there exists a bounded inverse operator $\mathfrak{N}^{-1}$ such that

$$\mathfrak{N}^{-1} : \mathfrak{R}_\beta^{l,2}W(\Pi) \longmapsto \mathfrak{D}_\beta^{l,2}W(\Pi), \tag{5.1}$$

with $\beta = |\beta_0|\sin\alpha$. The value $|\beta_0|$ is independent of α and depends on eigenvalues of the operator pencils associated with the corresponding linear problem (cf. [12]). Furthermore, the multidimensional space $\mathfrak{R}_\beta^{l,2}W(\Pi)$ to which the right-hand side of the linearized problem belongs can be described in a similar way as the space $\mathfrak{D}_\beta^{l,2}W(\Pi)$ (see above). Moreover, one is able to prove that there holds the inequality

$$\left\|\mathfrak{N}^{-1}\right\| \leqslant \frac{C}{\sin\alpha}, \tag{5.2}$$

where the constant C does not depend on α. Therefore, Problem (1.1)–(1.4) is equivalent to the subsequent operator equation in the space $\mathfrak{D}^{l,2}_{\beta}W(\Pi)$:

$$\boldsymbol{U} = \mathfrak{N}^{-1}\mathfrak{F}(\boldsymbol{U}) \equiv \mathfrak{K}(\boldsymbol{U}), \tag{5.3}$$

where $\boldsymbol{U} = (\boldsymbol{u}, q, \vartheta, \Psi_1, \Psi_2)^T$, and $\mathfrak{F}(\boldsymbol{U}) = (f_1(\boldsymbol{u}, q, \vartheta, \Psi_1, \Psi_2), f_2(\boldsymbol{u}, q, \vartheta, \Psi_1, \Psi_2), 0, f_3(\boldsymbol{u}, q, \vartheta, \Psi_1, \Psi_2), \dots)^T$ denotes the long vector of right-hand side after the linearization. The coordinates of the right-hand side vector depend on ε via the transformation mapping of the original flow domain. To prove the convergence of the successive approximations $(\boldsymbol{u}^n, q^n, \vartheta^n, \Psi_1^n, \Psi_2^n)^T$, it is sufficient to check that the operator $\mathfrak{K}$ is a contraction mapping in a ball of the space $\mathfrak{D}^{l,2}_{\beta}W(\Pi)$ for small values of ε and α.

Let us remark that the related isothermal problem to Problem (1.1)–(1.4) (i.e. without any inclusion of temperature) was analytically investigated in full detail in the papers [11, 12]. In order to prove Theorem 1 in details, one has to repeat and to modify all the considerations from those papers. Since the temperature equation is a nonlinear elliptic one, there are not essential changes in the proof. Thus we omit the detailed proof here.

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References

[1] F. Abergel and J.L. Bona. A mathematical theory for viscous free-surface flows over a perturbed plane. *Arch. Ration. Mech. Anal.*, **118**:71–93, 1992. http://dx.doi.org/10.1007/BF00375692.

[2] J.P. Burelbach, S.G. Bankhoff and S.H. Davis. Nonlinear stability of evaporating condensing liquid films. *J. Fluid Mech.*, **195**:463–494, 1988. http://dx.doi.org/10.1017/S0022112088002484.

[3] C. Cuvelier and J.M. Driessen. Thermocapillary free boundaries in crystal growth. *J. Fluid Mech.*, **169**:1–26, 1986. http://dx.doi.org/10.1017/S0022112086000526.

[4] I.V. Denisova and Š. Nečasova. Oberbeck–Boussinesq approximation for the motion of two incompressible fluids. *J. Math. Sci.*, **159**(4):436–451, 2009. http://dx.doi.org/10.1007/s10958-009-9455-6.

[5] P. Ehrhard and S.H. Davis. Non-isothermal spreading of liquid drops on horizontal plates. *J. Fluid Mech.*, **229**:365–388, 1991. http://dx.doi.org/10.1017/S0022112091003063.

[6] I.B. Erunova and V.Ya. Rivkind. Investigation of the problem of liquid evaporation. *Vestnik St. Petersburg Univ. Math.*, **24**:25–30, 1991.

[7] M.V. Lagunova. On the solvability of the plane problem of thermocapillary convection. *Probl. Mat. Anal. – Lin. and Nonlin. BVPs Spectr. Theory*, **10**:33–47, 1986. (In Russian)

[8] I.N. Larina, V.A. Rykov and E.M. Shakov. Evaporation from a surface and vapor flow through a plane channel into a vacuum. *Fluid Dynam.*, **31**:127–133, 1996. http://dx.doi.org/10.1007/BF02230757.

[9] S.A. Nazarov and K. Pileckas. On noncompact free boundary problems for the plane stationary Navier–Stokes equations. *J. Reine Angew. Math.*, **438**:103–141, 1993.

[10] P. Neittaanmäki and V.Ya. Rivkind. Mathematical modelling of liquid drops evaporation. In *Jyväskylä-St. Petersburg Seminar on Partial Differential Equations and Numerical Methods, Ber. Univ. Jyväskylä Math. Inst.*, volume 56, pp. 89–100, 1993.

[11] K. Pileckas and J. Socolowsky. Analysis of two linearized problems modeling viscous two-layer flows. *Math. Nachr.*, **245**:129–166, 2002.

[12] K. Pileckas and J. Socolowsky. Viscous two-fluid flows in perturbed unbounded domains. *Math. Nachr.*, **278**(5):589–623, 2005. http://dx.doi.org/10.1002/mana.200310260.

[13] V.V. Pukhnachov. Free boundary problems in the theory of thermo-capillary convection. In A. Bossavit, A. Damlamian and M. Fremond(Eds.), *Free Boundary Problems: Applications and Theory, vol. IV*, volume 121 of *Res. Notes Math.*, pp. 405–415. Pitman Advanced Publishing program, Boston, London, Melbourne, 1985.

[14] V.Ya. Rivkind. Numerical solution of coupled Navier–Stokes and Stefan equations. In P. Neittaanmäki(Ed.), *Proceedings of the Conference on Numerical Methods for Free Boundary Problems*, pp. 57–67. Birkhäuser Verlag, Basel, 1991.

[15] V.Ya. Rivkind and A.V. Ilin. On a numerical method for solving the nonisothermal flow problem of a liquid layer. *Numer. Methods*, **15**:98–104, 1988. (In Russian)

[16] P.A. Sackinger, R.A. Brown and J.J. Derby. A finite element method for analysis of fluid flow, heat transfer and free interfaces in Czochralski crystal growth. *Internat. J. Numer. Methods Fluids*, **9**:453–492, 1989. http://dx.doi.org/10.1002/fld.1650090406.

[17] G.R. Schmidt, T.J. Chung and A. Nadarajah. Thermocapillary flow with evaporation and condensation at low gravity. Part I: Non-deforming surface, and Part II: Deformable surface. *J. Fluid Mech.*, **294**:323–347, 349–366, 1995.

[18] Y. Shen, G.P. Neitzel, D.F. Jankowski and H.D. Mittelmann. Energy stability of thermocapillary convection in a model of the float-zone crystal-growth process. *J. Fluid Mech.*, **217**:639–660, 1990. http://dx.doi.org/10.1017/S002211209000088X.

[19] J. Socolowsky. Existence and uniqueness of the solution to a free boundary value problem with thermocapillary convection in an unbounded domain. *Acta Appl. Math.*, **37**:181–194, 1994. http://dx.doi.org/10.1007/BF00995140.

[20] J. Socolowsky. On the numerical solution of heat-conducting multiple-layer coating flows. *Lithuanian Math. J.*, **38**(1):98–115, 1998. http://dx.doi.org/10.1007/BF02465548.

[21] J. Socolowsky. The analysis of a coating flow with evaporation. In P. Neittaanmäki and L. Rivkind(Eds.), *Analysis and Approximation of Boundary Value Problems*, volume No. A 2/2000 of *Reports of the Department of Mathematical Information Technology Series A. Collections*, pp. 209–225. University of Jyväskylä, Finland, 2000.

[22] J. Socolowsky. On the Nusselt solution of a nonisothermal two-fluid inclined film flow. *Int. J. Math. Math. Sci. (IJMMS)*, p. 8 pages, 2009. http://dx.doi.org/10.1155/2009/981983. Electronic Journal

[23] V.A. Solonnikov. On the Stokes equation in domains with nonsmooth boundaries and on a viscous incompressible flow with a free surface. In *Nonlinear Partial Diff. Equations and Their Applications*, volume 3, pp. 340–423. College de France Seminar, 1980.

[24] V.A. Solonnikov. Solvability of the problem on the effluence of a viscous incompressible fluid into an open bassin. *Proc. Math. Inst. Steklov*, **179**(2):193–226, 1989.

[25] B.W. van de Fliert. A free boundary problem for evaporating layers. *Nonlinear Anal.*, **47**(3):1785–1796, 2001.
http://dx.doi.org/10.1016/S0362-546X(01)00310-8.