



A New Approach for Solving Nonlinear Singular Boundary Value Problems*

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Abstract. In this paper, an efficient method based on Quasi-Newton's method and the simplified reproducing kernel method is proposed for solving nonlinear singular boundary value problems. For the Quasi-Newton's method the convergence order is studied. The uniform convergence of the numerical solution as well as its derivatives are also proved. Numerical examples are given to demonstrate the efficiency and stability of the proposed method. The numerical results are compared with exact solutions and the outcomes of other existing numerical methods.

Keywords: nonlinear singular boundary value problem, numerical solution, Quasi-Newton's method, reproducing kernel method.

AMS Subject Classification: 34B16; 34K28; 47B32; 65L10.

1 Introduction

We now consider the following nonlinear singular boundary value problem with Neumann and Robin boundary conditions:

$$\begin{cases} u''(x) + \frac{\alpha}{x}u'(x) = f(x, u(x)), & 0 < x \leq 1, \quad \alpha \geq 1, \\ u'(0) = 0, \quad au(1) + bu'(1) = c. \end{cases} \quad (1.1)$$

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Here we assume nonlinear function $f(x, u)$ is continuous. $\frac{\partial f}{\partial u}$ exists and is continuous and $\frac{\partial f}{\partial u} \geq 0$ for any $0 \leq x \leq 1$.

Singular boundary value problems are frequent in a wide variety of applied mathematics and engineering practices, such as chemical reactions, gas dynamics, control and optimization theory, areas of biology and astrophysics [3, 13]. Due to the difficulty of singularity, it is usually impossible to obtain the analytical solution of such equations. Hence, it's quite necessary to seek accurate numerical solutions of equation (1.1). Taylor series method is applied and the error analysis is presented in [2]. The authors [6] apply He's variational iteration method for solving equation (1.1). In [7], cubic spline method to analyze equation (1.1) is presented. The authors [1] use B-spline functions to develop a numerical method for computing approximation to the solution of equation (1.1). A new approach implementing a modified decomposition method in combination with the cubic B-spline collocation technique is introduced in [8]. Furthermore, the improved Adomian decomposition based on Green's function is discussed [18]. The finite difference method and non-polynomial cubic spline method are given in [16] and [17] respectively. Recently, a novel approach based on the operational matrix of orthonormal Bernoulli polynomial has been proposed [13]. Numerical solution of equation (1.1) based on improved differential transform method (IDTM) has also been proposed by the authors in [10]. Furthermore, the similar method based on IDTM works well for the other type of nonlinear boundary value problems [11]. Although, these numerical methods are efficient and have many advantages, a lot of computational work or a high degree of smoothness are needed.

Recently, the theory of reproducing kernel has emerged as a powerful framework in numerical analysis, differential and integral equations, and probability and statistics [5]. Based on the reproducing kernel theory, the reproducing kernel method has been used and modified by many authors [4, 9, 14, 15, 19]. In this work, by combining with Newton iteration and modifying the reproducing kernel method, we will find the numerical solutions of equation (1.1). It's a different approach from those previous attempts. Our method can reduce computation cost and provide highly accurate global approximate solutions.

2 The Quasi-Newton's method

In this section, we describe the classical Newton method which is a very effective tool for solving nonlinear problems. Consider a nonlinear equation $f(x) = 0$, at the point $(x_0, f(x_0))$, we can get a tangent equation

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Furthermore, the equation $f(x) = 0$ can be approximated by the following equation

$$f(x_0) + f'(x_0)(x - x_0) = 0. \quad (2.1)$$

Then, a Newton's iterative formula for solving $f(x) = 0$ is defined as following:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots$$

Now we introduce the Fréchet derivative proposed by paper [12].

DEFINITION 1. Let $\mathcal{F} : X \rightarrow Y$, where X and Y are Banach spaces. Then a bounded and linear operator $\mathcal{A} : X \rightarrow Y$ is called a *Fréchet derivative* of \mathcal{F} at $u \in X$ if

$$\lim_{h \rightarrow 0} \frac{\|\mathcal{F}(u+h) - \mathcal{F}(u) - \mathcal{A}(h)\|}{\|h\|} = 0,$$

for all $h \in X$ denoted by $\mathcal{F}'(u)$.

Lemma 1. *If \mathcal{F} is a linear operator, there holds $\mathcal{F}'(u)=\mathcal{F}$, for all $u \in X$.*

Proof. Since \mathcal{F} is a linear operator, we have

$$\lim_{h \rightarrow 0} \frac{\|\mathcal{F}(u+h) - \mathcal{F}(u) - \mathcal{F}(h)\|}{\|h\|} = \lim_{h \rightarrow 0} \frac{\|\mathcal{F}(u) + \mathcal{F}(h) - \mathcal{F}(u) - \mathcal{F}(h)\|}{\|h\|} = 0.$$

Therefore, $\mathcal{F}'(u)=\mathcal{F}$. \square

Especially, we can obtain the following Lemma.

Lemma 2. *If \mathcal{F} is a constant operator, then $\mathcal{F}'(u)$ is a zero operator for all $u \in X$.*

Based on Fréchet derivative theory, we can get an approximate formula for equation (1.1) which is similar to formula (2.1).

Consider an operator $\mathcal{F} : C^2[0, 1] \rightarrow C[0, 1]$ as

$$\mathcal{F}(u) \triangleq xu'' + \alpha u' + x\mathcal{N}(u),$$

where $\mathcal{N}(u) = -f(x, u(x))$. Equation (1.1) can be rewritten as

$$\begin{cases} \mathcal{F}(u) = 0, & 0 < x \leq 1, \quad \alpha \geq 1, \\ u'(0) = 0, & au(1) + bu'(1) = c. \end{cases}$$

According the Definition (1) and Lemma (1), we can get the Fréchet derivative of \mathcal{F} at u_0 :

$$\mathcal{F}'(u_0) : u \mapsto xu'' + \alpha u' + x\mathcal{N}'(u_0)u. \tag{2.2}$$

Moreover the operator equation $\mathcal{F}(u) = 0$ can be approximated by

$$\mathcal{F}(u_0) + \mathcal{F}'(u_0)(u - u_0) = 0. \tag{2.3}$$

By substituting (2.2) into (2.3), we get the following linear equation

$$xu'' + \alpha u' + x\mathcal{N}'(u_0)(u - u_0) + x\mathcal{N}(u_0) = 0.$$

Therefore, we obtain a Quasi-Newton's iterative process as the following form

$$\begin{cases} xu''_{k+1} + \alpha u'_{k+1} + x\mathcal{N}'(u_k)u_{k+1} = x\mathcal{N}'(u_k)u_k - x\mathcal{N}(u_k), \\ u'_k(0) = 0, \quad au_k(1) + bu'_k(1) = c, \quad k = 0, 1, 2, \dots \end{cases} \tag{2.4}$$

and its general form

$$\mathcal{F}(u_k) + \mathcal{F}'(u_k)(u_{k+1} - u_k) = 0. \quad (2.5)$$

It can be seen equation (2.4) and equation (2.5) are linear equations. We could get the numerical solution. Firstly we choose a simple polynomial function satisfying the boundary conditions as an initial function u_0 . Then by substituting u_0 into equation (2.4), we can get a solution u_1 . And after several rounds of iteration, a high accuracy numerical result can be obtained. Next, we analyze the the convergence of our iterative algorithm.

Theorem 1. *Let Fréchet derivative \mathcal{F}' and \mathcal{N} be operators mentioned in equation (2.4) and equation (2.5). Assume that $\|\mathcal{F}'^{-1}\|\|\mathcal{N}''\| = \gamma < 1$. Further, the initial function u_0 satisfies that $\|u_0 - u\| < 1$, where u is the exact solution of equation (1.1). Then, iterative formula equation (2.4) or equation (2.5) is convergence, and the absolute error satisfies $|u_k - u| \leq \|u_k - u\| \leq \gamma^{2^k - 1}$.*

Proof. Based on equation (2.5), we have

$$u_{k+1} - u_k = -[\mathcal{F}'(u_k)]^{-1}\mathcal{F}(u_k).$$

Supposing $u(x)$ is a solution of $\mathcal{F}(u) = 0$, and $E_k(x) = u_k(x) - u(x)$. Then,

$$E_{k+1} = E_k - [\mathcal{F}'(u_k)]^{-1}(\mathcal{F}(u_k) - \mathcal{F}(u)).$$

Thus,

$$\mathcal{F}'(u_k)E_{k+1} = \mathcal{F}'(u_k)E_k - (\mathcal{F}(u_k) - \mathcal{F}(u)). \quad (2.6)$$

Note that

$$\mathcal{F}'(u_k)E_k = xE_k'' + \alpha E_k' + x\mathcal{N}'(u_k)E_k, \quad (2.7)$$

$$\begin{aligned} \mathcal{F}(u_k) - \mathcal{F}(u) &= xu_k'' + \alpha u_k' + x\mathcal{N}(u_k) - (xu'' + \alpha u' + x\mathcal{N}(u)) \\ &= xE_k'' + \alpha E_k' + x(\mathcal{N}(u_k) - \mathcal{N}(u)) = xE_k'' + \alpha E_k' + x\mathcal{N}'(\xi_k)E_k. \end{aligned} \quad (2.8)$$

Substituting (2.7), (2.8) into (2.6), we obtain

$$\begin{aligned} \mathcal{F}'(u_k)E_{k+1} &= xE_k'' + \alpha E_k' + x\mathcal{N}'(u_k)E_k - (xE_k'' + \alpha E_k' + x\mathcal{N}'(\xi_k)E_k) \\ &= x(\mathcal{N}'(u_k) - \mathcal{N}'(\xi_k))E_k = x\mathcal{N}''(\eta_k)(u_k - \xi_k)E_k. \end{aligned}$$

As $|u_k - \xi_k| \leq |u_k - u| = |E_k|$, we have

$$\|E_{k+1}\| \leq \|\mathcal{F}'(u_k)^{-1}\|\|\mathcal{N}''(\eta_k)\|\|E_k\|^2 < \gamma\|E_k\|^2.$$

Furthermore, we discuss the convergence order based on the mathematical induction. Firstly,

$$\|E_0\| = \|u_0 - u\| < 1 = \gamma^{2^0 - 1}.$$

Let

$$\|E_k\| = \|u_k - u\| = \gamma^{2^k - 1},$$

Hence,

$$\|E_{k+1}\| < \gamma\|E_k\|^2 = \gamma(\gamma^{2^k - 1})^2 = \gamma^{2^{k+1} - 1}.$$

Thus, the proof is completed. \square

3 The simplified reproducing kernel method

In this section, we focus on solving the linear equation by simplified reproducing kernel method. For convenience, we discuss the simplified equation of equation (2.4) as follows

$$\begin{cases} xu''(x) + \alpha u'(x) + xa_2(x)u = h(x), \\ u'(0) = 0, \quad au(1) + bu'(1) = c, \end{cases} \tag{3.1}$$

where $a_2(x) = \mathcal{N}'(u_k)$, $h(x) = x(\mathcal{N}'(u_k)u_k - \mathcal{N}(u_k))$.

Now, we introduce the reproducing kernel method for equation (3.1). From [4], it follows that $W_2^3[0, 1]$, $W_2^1[0, 1]$ are reproducing kernel spaces with reproducing kernel functions $R_t(x)$ and $r_t(x)$ respectively. Considering an operator $\mathcal{L} : W_2^3[0, 1] \rightarrow W_2^1[0, 1]$,

$$\mathcal{L}u = xu''(x) + \alpha u'(x) + xa_2(x)u(x),$$

equation (3.1) can be written as

$$\begin{cases} \mathcal{L}u = h(x), \\ u'(0) = 0, \quad au(1) + bu'(1) = c. \end{cases} \tag{3.2}$$

Lemma 3. Let \mathcal{L}^* be a conjugate operator of \mathcal{L} , then $(\mathcal{L}^*r_t)(x) = (\mathcal{L}R_x)(t)$.

Taking a dense set $\{x_i\}_{i=1}^\infty$ on $[0, 1]$, we get a function system $\{\psi_i(x)\}_{i=1}^\infty$ in W_2^3 , where $\psi_i(x) = \mathcal{L}^*r_x(x_i)$.

Theorem 2. $\{\psi_i(x)\}_{i=1}^n$ is linear independent.

For the proof, one may refer to [4].

Let

$$\begin{aligned} \varphi_1(x) &= \left. \frac{\partial R_t(x)}{\partial t} \right|_{t=0}, \quad \varphi_2(x) = aR_1(x) + b \left. \frac{\partial R_t(x)}{\partial t} \right|_{t=1}, \\ S_{n+2} &= \text{span}\{\{\psi_i\}_1^n \cup \varphi_1 \cup \varphi_2\}, \quad S = \overline{\bigcup_{i=1}^\infty S_{i+2}}. \end{aligned}$$

Then $S_{n+2} \subset S \subset W_2^3$.

Theorem 3. Let u be the exact solution of the operator equation (3.2). Then $\mathcal{P}u$ is also the solution of (3.2), where \mathcal{P} is the orthogonal project operator from W_2^3 to S .

Let \mathcal{P}_{n+2} is the orthogonal project operator from W_2^3 to S_{n+2} .

Theorem 4. $\mathcal{P}_{n+2}u$ is the solution of the following linear equation system

$$\begin{cases} \langle v, \psi_i \rangle = h(x_i), \quad i = 1, 2, \dots, n, \\ \langle v, \varphi_1 \rangle = 0, \quad \langle v, \varphi_2 \rangle = c. \end{cases} \tag{3.3}$$

Proof.

$$\begin{aligned}
\langle \mathcal{P}_{n+2}u(\cdot), \psi_i(\cdot) \rangle &= \langle u(\cdot), \mathcal{P}_{n+2}\psi_i(\cdot) \rangle = \langle u(\cdot), \psi_i(\cdot) \rangle = \langle u(\cdot), \mathcal{L}^*r_{x_i}(\cdot) \rangle \\
&= \langle \mathcal{L}u(\cdot), r_{x_i}(\cdot) \rangle = \langle h(\cdot), r_{x_i}(\cdot) \rangle = h(x_i), \\
\langle \mathcal{P}_{n+2}u(\cdot), \varphi_1(\cdot) \rangle &= \langle u(\cdot), \mathcal{P}_{n+2}\varphi_1(\cdot) \rangle = \langle u(\cdot), \varphi_1(\cdot) \rangle = \left\langle u(\cdot), \frac{\partial R_x(\cdot)}{\partial x} \Big|_{x=0} \right\rangle \\
&= \frac{\partial}{\partial x} \langle u(\cdot), R_x(\cdot) \rangle \Big|_{x=0} = \frac{\partial}{\partial x} u(x) \Big|_{x=0} = u'(0) = 0, \\
\langle \mathcal{P}_{n+2}u(\cdot), \varphi_2(\cdot) \rangle &= \langle u(\cdot), \mathcal{P}_{n+2}\varphi_2(\cdot) \rangle = \langle u(\cdot), \varphi_2(\cdot) \rangle \\
&= \left\langle u(\cdot), aR_1(\cdot) + b \frac{\partial R_x(\cdot)}{\partial x} \Big|_{x=1} \right\rangle = au(1) + b \frac{\partial}{\partial x} \langle u(\cdot), R_x(\cdot) \rangle \Big|_{x=1} \\
&= au(1) + b \frac{\partial}{\partial x} u(x) \Big|_{x=1} = au(1) + bu'(1) = c.
\end{aligned}$$

□

Denote $u_n = \mathcal{P}_{n+2}u$. It can be shown that u_n is an approximate solution of u . Furthermore, we can prove the uniform convergence.

Theorem 5. $u_n^{(i)}(x)$ uniformly converge to $u^{(i)}(x)$ on $[0, 1]$, $i = 0, 1, 2$.

Proof.

$$\begin{aligned}
|u_n^{(i)}(x) - u^{(i)}(x)| &= |\langle u_n(\cdot) - u(\cdot), R_x(\cdot) \rangle^{(i)}| \\
&= \left| \left\langle u_n(\cdot) - u(\cdot), \frac{\partial^i}{\partial x^i} R_x(\cdot) \right\rangle \right| \leq \|u_n - u\| \left\| \frac{\partial^i R_x}{\partial x^i} \right\|.
\end{aligned}$$

Note that $\left\| \frac{\partial^i R_x}{\partial x^i} \right\|$ is bounded on $[0, 1]$, thus

$$|u_n^{(i)}(x) - u^{(i)}(x)| \leq M \|u_n - u\| \rightarrow 0.$$

The proof is completed. □

Hence, u_n is a good approximate solution of (3.2). Since $u_n \in S_{n+2}$, we get

$$u_n = \sum_{i=1}^n \alpha_i \psi_i + \beta_1 \varphi_1 + \beta_2 \varphi_2. \quad (3.4)$$

As u_n is the solution of equation (3.3), we have

$$\begin{cases} \sum_{j=1}^n \alpha_j \langle \psi_j, \psi_i \rangle + \beta_1 \langle \varphi_1, \psi_i \rangle + \beta_2 \langle \varphi_2, \psi_i \rangle = h(x_i), & i = 1, 2, \dots, n, \\ \sum_{j=1}^n \alpha_j \langle \psi_j, \varphi_1 \rangle + \beta_1 \langle \varphi_1, \varphi_1 \rangle + \beta_2 \langle \varphi_2, \varphi_1 \rangle = 0, \\ \sum_{j=1}^n \alpha_j \langle \psi_j, \varphi_2 \rangle + \beta_1 \langle \varphi_1, \varphi_2 \rangle + \beta_2 \langle \varphi_2, \varphi_2 \rangle = c. \end{cases} \quad (3.5)$$

The coefficients of formula (3.4) are the solution of linear equation system (3.5), hence we get the approximate solution u_n of equation (3.1).

4 Numerical examples

In this section, numerical examples are tested to demonstrate the efficiency and stability of our method. All examples are computed by Mathematica 7.0.

Example 1. Consider the nonlinear differential equation [1, 18]:

$$\begin{cases} u''(x) + \frac{1}{x}u'(x) = -e^{u(x)}, & 0 < x < 1, \\ u'(0) = 0, u(1) = 0. \end{cases}$$

The exact solution is $u(x) = 2 \ln \left(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})x^2+1} \right)$. In our method, we need to take a group of parameters, such as initial function $u_0(x)$, the number of iterations k , and the number of dense points n . Let $u_0(x) = x^4$. The absolute error function is $E = |u_{k,n}(x) - u(x)|$, $0 < x \leq 1$, where $u_{k,n}(x)$ is the approximate solution. Table 1 compares the results for this example by the proposed method and the methods in [1, 18].

Table 1. Maximum absolute error for Example 1

k	n	E	n	E in [18]	n	E in [1]
3	n=4	5.67E-05	12	4.01E-04	20	3.16E-05
3	n=6	3.14E-06	16	2.54E-05	40	7.87E-06
3	n=8	6.35E-07	20	2.10E-06	60	3.50E-06

Figure 1 shows comparison of exact and approximate solution for $k = 3$, $n = 8$ and also plots the absolute errors for $k = 3$, $n = 4, 6, 8$. We can see that the presented method converges rapidly to the exact solution by iterating only three times.

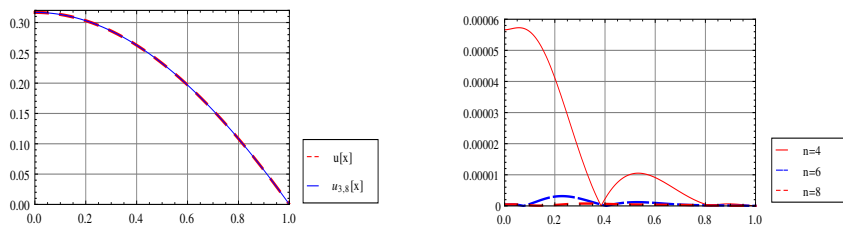


Figure 1. Comparison of exact solution $u(x)$ and approximate solution $u_{3,8}(x)$ (left side) and E (right side) for Example 1($k = 3$).

Example 2. Consider the nonlinear singular boundary value problem describing the equilibrium of the isothermal gas sphere [3, 10, 18]

$$\begin{cases} u''(x) + \frac{2}{x}u'(x) = -u^5(x), & 0 < x < 1, \\ u'(0) = 0, u(1) = \sqrt{3/4}. \end{cases}$$

The exact solution is $u(x) = \sqrt{3/(3+x^2)}$. Let $u_0(x) = x^2$. Table 2 shows the comparisons of the presented method and the method used in [3, 10, 18]. After three iterations, the maximal absolute error can reach 10^{-7} at $n=10$. We have higher accuracy compared to the other mentioned methods.

Figure 2 gives comparison of $u(x)$ and $u_{4,10}(x)$ and also shows the absolute errors for $k = 3, n = 4, 6, 10$. It indicates that the curves of the exact solution and the approximate solution are overlap. The approximate solution is getting more and more accurate as the number of dense points n increases. Our approach has good convergence and precision for numerical calculation.

Table 2. Maximum absolute error for Example 2

k	n	E	n	E in [18]	n	E in [3]	n	E in [10]
4	$n = 4$	3.25E-05	12	1.40E-03	16	3.64E-04	12	1.68E-04
4	$n = 6$	6.74E-06	16	2.47E-04	32	2.49E-05	16	1.65E-05
4	$n = 10$	7.30E-07	20	4.86E-05	64	1.60E-06	20	1.66E-06

Figure 2 gives comparison of $u(x)$ and $u_{4,10}(x)$ and also shows the absolute errors for $k = 3, n = 4, 6, 10$. It indicates that the curves of the exact solution and the approximate solution are overlap. The approximate solution is getting more and more accurate as the number of dense points n increases. Our approach has good convergence and precision for numerical calculation.

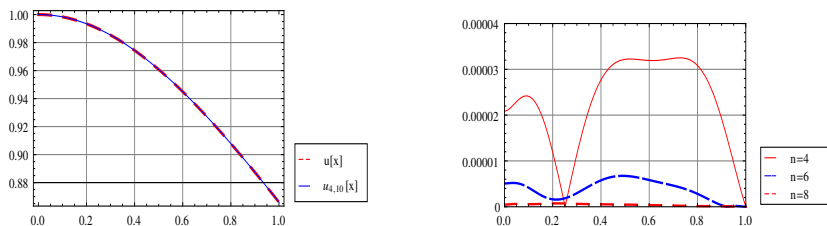


Figure 2. Comparison of exact solution $u(x)$ and approximate solution $u_{4,10}(x)$ (left side) and E (right side) for Example 2 ($k = 4$).

Example 3. Consider the nonlinear singular boundary value problem studying the distribution of heat sources in the human head [7, 10, 13, 16].

$$\begin{cases} u''(x) + \frac{2}{x}u'(x) = -e^{-u(x)}, & 0 < x \leq 1, \\ u'(0) = 0, & u(1) + u'(1) = 0. \end{cases} \quad (4.1)$$

Let $u_0(x) = x^3$. Since the exact solution of equation (4.1) is not known, we calculate function values of approximate solution in ten points which are presented in [7, 10, 13, 16]. The numerical results for $k = 3, n = 10$ are listed in Table 3. The proposed method is in good agreement to other methods [7, 10, 13, 16].

Table 3. Numerical results for Example 3

x	Method in [16]	Method in [7]	Method in [13]	Method in [10]	Present Method
0	0.3675169710	0.3675179806	0.3675152742	0.3675167997	0.3675166189
0.1	0.3663623697	0.3663634922	0.3663623292	0.3663623137	0.3663622527
0.2	0.3628941066	0.3628952219	0.3628940661	0.3628940507	0.3628940189
0.3	0.3570975862	0.3570986892	0.3570975457	0.3570975301	0.3570975087
0.4	0.3489484612	0.3489495463	0.3489484206	0.3489484049	0.3489483815
0.5	0.3384121893	0.3384132503	0.3384121487	0.3384121330	0.3384121172
0.6	0.3254435631	0.3254445926	0.3254435224	0.3254435063	0.3254434945
0.7	0.3099860810	0.3099870706	0.3099860402	0.3099860240	0.3099860196
0.8	0.2919711440	0.2919720836	0.2919711030	0.2919710864	0.2919711009
0.9	0.2713170512	0.2713179290	0.2713170101	0.2713169936	0.2713170010
1.0	0.2479277646	0.2479285660	0.2479277233	0.2479277073	0.2479277424

Example 4. Consider the nonlinear singular boundary value problem arising in the study of steady-state oxygen diffusion in a spherical cell [1, 10, 13, 17].

$$\begin{cases} u''(x) + \frac{\alpha}{x}u'(x) = \frac{\delta u(x)}{u(x)+\mu}, & \delta > 0, \mu > 0, \\ u'(0) = 0, \quad 5u(1) + 5u'(1) = 5, \end{cases} \tag{4.2}$$

where δ and μ are often taken as 0.76129 and 0.03119, respectively. And we take the value of α as 2. This equation does not have exact solution. Instead, we discuss the absolute residual error function which is a measure of how well the approximation satisfies the equation (4.2) as

$$R = \left| u''_{k,n} + \frac{\alpha}{x}u'_{k,n} - \frac{\delta u_{k,n}}{u_{k,n} + \mu} \right|, \quad 0 < x \leq 1.$$

Taking $u_0(x) = x^3$, a good approximation is obtained for $k = 4, n = 10$. The numerical results of different methods are listed in Table 4.

Table 4. Numerical results for Example 4

x	Method in [17]	Method in [1]	Method in [13]	Method in [10]	Present Method
0	0.8284833089	0.8284832730	0.8284832819	0.8284832870	0.8284833754
0.1	0.8297060779	0.8297060752	0.8297060924	0.8297060890	0.8297061155
0.2	0.8333747471	0.8333747169	0.8333747336	0.8333747303	0.8333747353
0.3	0.8394899001	0.8394898981	0.8394899140	0.8394899106	0.8394899068
0.4	0.8480527729	0.8480527704	0.8480527850	0.8480527816	0.8480527759
0.5	0.8590649189	0.8590649140	0.8590649272	0.8590649239	0.8590649167
0.6	0.8725283096	0.8725283084	0.8725283200	0.8725283166	0.8725283100
0.7	0.8884452969	0.8884452959	0.8884453056	0.8884453023	0.8884452975
0.8	0.9068185417	0.9068185403	0.9068185481	0.9068185448	0.9068185422
0.9	0.9276509838	0.9276509825	0.9276509884	0.9276509853	0.9276509849
1.0	0.9509457969	0.9509457946	0.9509457985	0.9509457960	0.9509457969

Table 5 describes the comparison of absolute residual error between presented method and IDTM [10]. The implementation of our approach in good line with existing methods [1, 10, 13, 17] has a fast convergence speed and high accuracy.

Table 5. Maximum absolute residual error for Example 4

k	n	R	n	R in [10]
4	$n = 4$	4.25E-05	4	4.84E-04
4	$n = 6$	6.14E-06	6	6.78E-05
4	$n = 8$	1.38E-06	8	9.45E-06
4	$n = 10$	8.39E-07	10	1.31E-06

5 Conclusions

To summarize, in this paper, we propose a new approach for solving nonlinear singular boundary value problem. First, we reduce the nonlinear problem to a linear problem based on the Quasi-Newton's method. And second, the simplified reproducing kernel method which has less computation than before is presented to solve the linear problem. The approximate solution can be obtained after iterative computation. Our method is tested by four examples arising in applied science. The numerical results show that our method has higher accuracy, less calculation and high efficiency. It is worthy to note that the approach detailed here can be readily adapted to the case of nonlinear integral and integro-differential equation with various boundary value conditions.

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