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Nonstationary heat equation with nonlinear side condition

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Abstract. The initial boundary value problem for the nonstationary heat equation is studied in a bounded domain with the specific overdetermination condition. This condition is nonlinear and can be interpreted as the energy functional. In present paper we construct the class of solutions to this problem.

Keywords: nonstationary heat equation; inverse problem; very weak solution; nonlinear side condition.

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1 Introduction

Let us start with the nonstationary boundary value problem to the heat equation

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ u(x,t)|_{\partial\Omega\times[0,T]} = 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a simply connected bounded domain, the boundary $\partial\Omega$ is C^2 smooth, f is the internal source that heats or cools the system, u is the temperature and u_0 is the initial temperature. For the case when functions f, u_0 are prescribed and u is the unknown function, we have the classical initial boundary value problem for heat equation. The unique solvability of this problem is standard and well-established (see, for example, [8]).

There is an amount of papers where some additional integral condition

$$\int_\Omega u(x,t)\mathrm{d}x = F(t), \quad F(0) = \int_\Omega u_0(x)\mathrm{d}x \tag{1.2}$$

is prescribed (see, e.g., [2, 3, 4, 6, 9, 10, 11, 12, 13, 14]). Then, the solution of problem (1.1)–(1.2) is a pair of functions u and f. In other words, problem

(1.1), (1.2) can be seen as an inverse problem where we prescribe the time dependent function F. Inverse problems were studied by many mathematicians, starting with the works of J. R. Cannon (see [3, 4]) and then by the others (see [6, 11, 12, 13, 16]). However, in all mentioned papers, problem (1.1)–(1.2) was considered under assumption that function $F(t)$ is sufficiently smooth, for example, assuming that the derivative $F'(t)$ exists. Nevertheless, in recent papers (see [7,15]) problem (1.1)–(1.2) was studied under the minimal regularity of function $F(t)$, i.e., assuming that $F \in L^2(0,T)$.

In some papers, for example in [3, 4], integral (1.2) is called an energy[1]. However, in this paper instead of the linear side condition (1.2) we consider the nonlinear side condition:

$$\int_\Omega u^2(x,t)\mathrm{d}x = E^2(t), \quad E(0) = \|u_0\|_{L^2(\Omega)}.$$

It can be interpreted as the energy functional for the heat equation and it measures the distance (in L^2-norm) from the trivial equilibrium solution $u = 0$. This expression also reminds the elastic potential energy of a spring.

2 Notation and auxiliary results

In this paper, we will use the following notation. If G is the domain in $\mathbb{R}^n$, $C^\infty(G)$ means, as usual, the set of all infinitely differentiable functions in G and $C_0^\infty(G)$ is the subset of functions from $C^\infty(G)$ with compact supports in G. The space $C^m(\overline{G})$ (m is a nonnegative integer number) consists of m times continuously differentiable functions in $\overline{G}$ with the norm

$$\|u\|_{C^m(\overline{G})} = \sum_{|\alpha|=0}^{m} \sup_{x\in\overline{G}} |D^\alpha u(x)|.$$

For nonnegative integer l and $q > 0$ we use the usual notation for Lebesgue $L^q(G)$ and Sobolev $W^{l,q}(G)$ spaces with the norms

$$\|u\|_{L^q(G)} = \Big(\int_G |u(x)|^q dx\Big)^{1/q}, \quad \|u\|_{W^{l,q}(G)} = \Big(\sum_{|\alpha|=0}^{l}\int_G |D^\alpha u(x)|^q dx\Big)^{1/q}.$$

$W^{l-1/q,q}(\partial G)$ is the trace space on ∂G of functions from $W^{l,q}(G)$. The space $\mathring{W}^{1,2}(G)$ is the closure of $C_0^\infty(G)$ in the norm of $W^{1,2}(G)$ (see [1,8]).
The space $W^{-2,2}(G)$ denotes the dual space of $W^{2,2}(G)\cap\mathring{W}^{1,2}(G)$ with pairing $< h,\zeta >_G$ for any functional $h \in W^{-2,2}(G)$ and test function $\zeta \in W^{2,2}(G) \cap \mathring{W}^{1,2}(G)$. The norm in $W^{-2,2}(G)$ is defined in a usual way:

$$\|h\|_{W^{-2,2}(G)} = \sup_{\zeta\in W^{2,2}(G)\cap\mathring{W}^{1,2}(G)} \frac{|< h,\zeta >|}{\|\zeta\|_{W^{2,2}(G)}}.$$

The space $W^{-1/2,2}(\partial G)$ denotes the dual space of $W^{1/2,2}(\partial G)$ with pairing $< g,\xi >_{\partial G}$ for any functional $g \in W^{-1/2,2}(\partial G)$ and function $\xi \in W^{1/2,2}(\partial G)$.

[1] Notice that integral (1.2) does not actually describe ”energy” in the physical sense.

The norm of an element u in the function space V is denoted by $\|u\|_V$. Then, $L^2(0,T;V)$ is the space of functions u, depending on the space variable x and time variable t, such that $u(\cdot,t) \in V$ for almost all $t \in [0,T]$ and the norm

$$\|u\|_{L^2(0,T;V)} = \Big(\int_0^T \|u(\cdot,t)\|_V^2 dt\Big)^{1/2}$$

is finite.

Lemma 1. *Let G is a bounded domain in $\mathbb{R}^n$ and ∂G is C^2-smooth. Let $v_k(x) \in W^{2,2}(G) \cap \mathring{W}^{1,2}(G)$ and numbers λ_k are eigenfunctions and eigenvalues of the Laplace operator:*

$$\begin{cases} -\Delta v_k(x) = \lambda_k v_k(x), & x \in G, \\ v_k(x)\big|_{\partial G} = 0. \end{cases}$$

Then, $\lambda_k > 0$ and $\lim\limits_{k\to\infty} \lambda_k = \infty$. The eigenfunctions $v_k(x)$ are orthogonal in $L^2(G)$ and we assume that $v_k(x)$ are normalized in $L^2(G)$, i.e.,

$$\int_G v_k(x) v_l(x)\, dx = \delta_{lk} = \begin{cases} 1, & l = k, \\ 0, & l \neq k. \end{cases}$$

Moreover,

$$\int_G \nabla v_k(x) \cdot \nabla v_l(x)\, dx = \lambda_k \delta_{lk} = \begin{cases} \lambda_k, & l = k, \\ 0, & l \neq k. \end{cases}$$

For the details see [8].

Lemma 2. *Let G be a bounded domain in $\mathbb{R}^n, n \geq 1$, $\{v_k(x)\}$ be a basis in Hilbert space $\mathring{W}^{1,2}(G)$ and $h(x) = \sum\limits_{k=1}^{\infty} h_k v_k(x)$.*

1. *If $\sum\limits_{k=1}^{\infty} \dfrac{h_k^2}{1+\lambda_k^2} < \infty$, where λ_k is an eigenvalue corresponding eigenfunction $v_k(x)$, then $\displaystyle\int_G h(x)\eta(x)\, \mathrm{d}x$ is a bounded functional in $W^{2,2}(G)$.*
2. *If $H(\eta)$ be a bounded functional in $W^{2,2}(G)$, i.e. $H(\eta) \in W^{-2,2}(G)$, then $H(\eta) = \int_G h(x)\eta(x)\, \mathrm{d}x$ and $\sum\limits_{k=1}^{\infty} \dfrac{h_k^2}{1+\lambda_k^2} < \infty$ for any $\eta \in W^{2,2}(G)$.*

Proof.

1. Using the properties of the eigenfunctions and the Cauchy–Schwarz inequality we get

$$\int_G h(x)\eta(x)\, \mathrm{d}x = \lim_{N\to\infty} \int_G \sum_{k=1}^N h_k v_k(x) \sum_{k=1}^N \eta_k v_k(x)\, \mathrm{d}x = \lim_{N\to\infty} \sum_{k=1}^N h_k \eta_k$$
$$\leq \lim_{N\to\infty} \Big(\sum_{k=1}^N \frac{h_k^2}{1+\lambda_k^2}\Big)^{1/2} \Big(\sum_{k=1}^N \eta_k^2 (1+\lambda_k^2)\Big)^{1/2} \leq c\Big(\sum_{k=1}^\infty \eta_k^2 (1+\lambda_k^2)\Big)^{1/2}$$
$$= c\|\eta(x)\|_{W^{2,2}(G)}.$$

2. Let us denote $h_k = H(v_k(x))$. Then,

$$H(\eta(x)) = \lim_{N\to\infty} H\Big(\sum_{k=1}^{N} \eta_k v_k(x)\Big) = \lim_{N\to\infty} \sum_{k=1}^{N} \eta_k H(v_k(x)) = \lim_{N\to\infty} \sum_{k=1}^{N} \eta_k h_k$$
$$= \sum_{k=1}^{\infty} \eta_k h_k = \int_G h(x)\eta(x)\,\mathrm{d}x,$$

where $h(x) = \sum\limits_{k=1}^{\infty} h_k v_k(x)$.

Next, we prove that $\sum\limits_{k=1}^{\infty} \dfrac{h_k^2}{1+\lambda_k^2} < \infty$. Since $H(\eta)$ is a functional in $W^{2,2}(G)$, we have that

$$|H(\eta)| \le c\|\eta\|_{W^{2,2}(G)},$$

i.e.,

$$\sum_{k=1}^{\infty} h_k\eta_k \le c\|\eta\|_{W^{2,2}(G)} \le c\Big(\sum_{k=1}^{\infty} \eta_k^2(1+\lambda_k^2)\Big)^{1/2}. \tag{2.1}$$

Let us take

$$\eta_k = \begin{cases} h_k/(1+\lambda_k^2), & k \le N, \\ 0, & k > N. \end{cases} \tag{2.2}$$

Substituting (2.2) into (2.1) we obtain

$$\Big|\sum_{k=1}^{\infty} h_k\eta_k\Big| = \Big|\sum_{k=1}^{N} \frac{h_k^2}{1+\lambda_k^2}\Big| \le c\Big(\sum_{k=1}^{N} \frac{h_k^2}{(1+\lambda_k^2)^2}(1+\lambda_k^2)\Big)^{1/2}$$
$$\le c\Big(\sum_{k=1}^{N} \frac{h_k^2}{1+\lambda_k^2}\Big)^{1/2},$$

i.e., $\Big|\sum\limits_{k=1}^{N} \dfrac{h_k^2}{1+\lambda_k^2}\Big| \le c\Big(\sum\limits_{k=1}^{N} \dfrac{h_k^2}{1+\lambda_k^2}\Big)^{1/2}$.

Dividing both sides by $\Big(\sum\limits_{k=1}^{N} \dfrac{h_k^2}{1+\lambda_k^2}\Big)^{1/2}$ we get $\Big(\sum\limits_{k=1}^{N} \dfrac{h_k^2}{1+\lambda_k^2}\Big)^{1/2} \le c$. Since constant c in the last estimate does not depend on N, we can pass to a limit as $N \to \infty$ and we obtain:

$$\sum_{k=1}^{\infty} \frac{h_k^2}{1+\lambda_k^2} < \infty.$$

□

Remark 1. If function h depends on time variable t and space variable x, Lemma 2 remains valid with only difference that h_k depends on t.

Remark 2. Notice that $\|H\|_{W^{-2,2}(G)} \sim \sum\limits_{k=1}^{\infty} \dfrac{h_k^2}{1+\lambda_k^2}$.

3 Formulation of problem and main result

In a bounded simply connected domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with C^2 smooth boundary $\partial\Omega$ we consider

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ u(x,t)|_{\partial\Omega\times[0,T]} = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{3.1}$$

with additionally prescribed nonlinear side condition

$$\int_\Omega u^2(x,t)\mathrm{d}x = E^2(t), \ E(0) = \|u_0\|_{L^2(\Omega)}, \tag{3.2}$$

where u and f are unknown functions while E and u_0 are given functions.

Definition 1. The pair $(u(x,t), f(x,t))$ with functions $u \in L^2(0,T; L^2(\Omega))$, $u_t \in L^2(0,T; L^2(\Omega))$ and $f \in L^2(0,T; W^{-2,2}(\Omega))$ is called a very weak solution of problem (3.1)–(3.2) if the function u satisfies the initial condition $u(x,0) = u_0(x)$, the pair $(u(x,t), f(x,t))$ satisfies the integral identity for any $\eta \in L^2(0,T; W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega))$

$$\int_0^T \int_\Omega u_t(x,t)\eta(x,t)\mathrm{d}x - \int_0^T \int_\Omega u(x,t)\Delta\eta(x,t)\mathrm{d}x = \int_0^T \int_\Omega f(x,t)\eta(x,t)\mathrm{d}x$$

and u satisfies the nonlinear side condition (3.2).

Deriving the definition of a very weak solution, we multiplied the heat equation $(3.1)_1$ by the test function $\eta \in L^2(0,T; W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega))$ and then we integrated twice by parts over Ω the second term on the left-hand side. Doing this we got two integrals over the boundary $\partial\Omega$:

$$\int_{\partial\Omega} (\nabla u \cdot \mathbf{n})\eta \, dS, \quad \int_{\partial\Omega} u(\nabla\eta \cdot \mathbf{n}) \, dS,$$

where $\mathbf{n}$ is a unit vector of the outward normal to $\partial\Omega$.
Since $\eta \in L^2(0,T; W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega))$, i.e., $\eta=0$ on the boundary $\partial\Omega$ in the trace sense, the integral $\int_{\partial\Omega}(\nabla u \cdot \mathbf{n})\eta \, dS$ is equal to zero. The integral $\int_{\partial\Omega} u(\nabla\eta \cdot \mathbf{n}) \, dS$ must be understood as the functional $u \in W^{-1/2,2}(\partial\Omega)$ applied to the test function $\nabla\eta \in W^{1/2,2}(\partial\Omega)$. Indeed, since $\eta \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$, we have $\nabla\eta \in W^{1,2}(\Omega)$ and $\nabla\eta \in W^{1/2,2}(\partial\Omega)$ (see [1]). This implies that the boundary condition $(3.1)_2$ yields $\int_{\partial\Omega} u(\nabla\eta \cdot \mathbf{n}) \, dS = 0$.

The main result of this paper is formulated in the following theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded simply connected domain, the boundary $\partial\Omega$ is C^2 smooth, initial function $u_0 \in L^2(\Omega)$ and function $E \in W^{1,2}(0,T)$, $E(0) = \|u_0\|_{L^2(\Omega)}$. Then, there exists at least one very weak solution of problem* (3.1)–(3.2).

4 Proof of the main result

We look for the approximate solution in the form:

$$u^{(N)}(x,t) = \sum_{k=1}^{N} w_k^{(N)}(t) v_k(x), \quad f^{(N)}(x,t) = \sum_{k=1}^{N} q_k^{(N)}(t) v_k(x), \tag{4.1}$$

where $v_k(x)$ are eigenfunctions of the Laplace operator.

Functions $w_k^{(N)}(t)$ and $q_k^{(N)}(t)$ can be found from the following system:

$$\begin{cases} \int_\Omega u_t^{(N)}(x,t)\, v_k(x) \mathrm{d}x - \int_\Omega u^{(N)}(x,t)\, \Delta v_k(x) \mathrm{d}x = \int_\Omega f^{(N)}(x,t)\, v_k(x) \mathrm{d}x, \\ u^{(N)}(x,0) = \sum\limits_{k=1}^{N} \beta_k v_k(x), \\ \int_\Omega |u^{(N)}(x,t)|^2 \mathrm{d}x = \dfrac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum\limits_{k=1}^{N} \beta_k^2\, E^2(t), \end{cases} \tag{4.2}$$

where $\beta_k, \quad k = 1, ..., N$, are the Fourier coefficients of $u_0(x)$.

Equality $(4.2)_1$ and initial condition $(4.2)_2$ yields the following problem:

$$\begin{cases} \left(w_l^{(N)}(t)\right)_t' + \lambda_l\, w_l^{(N)}(t) = q_l^{(N)}(t), \\ w_l^{(N)}(0) = \beta_l. \end{cases} \tag{4.3}$$

For all $l = 1, 2, ..., N$ the solution of (4.3) is:

$$w_l^{(N)}(t) = \int_0^t e^{-\lambda_l\,(t-\tau)}\, q_l^{(N)}(\tau) \mathrm{d}\tau + \beta_k. \tag{4.4}$$

Substituting (4.1) into the nonlinear condition $(4.2)_3$ and using the orthogonality properties of the eigenfunctions v_l (see Lemma 1) we get

$$\begin{aligned} &\int_\Omega |u^{(N)}(x,t)|^2 \mathrm{d}x = \int_\Omega \Big|\sum_{l=1}^{N} w_l^{(N)}(t) v_l(x)\Big|^2 \mathrm{d}x \\ &= \sum_{l=1}^{N} \left(w_l^{(N)}(t)\right)^2 \int_\Omega v_l^2(x) \mathrm{d}x = \sum_{l=1}^{N} \left(w_l^{(N)}(t)\right)^2 = \sum_{l=1}^{N} \gamma_l^2\, E^2(t), \end{aligned} \tag{4.5}$$

where $\sum_{l=1}^{\infty} \gamma_l^2 = 1$.

In order to satisfy condition (4.5) we choose that

$$\left(w_l^{(N)}(t)\right)^2 = \gamma_l^2\, E^2(t), \quad \text{i.e.,} \quad w_l^{(N)}(t) = \gamma_l\, E(t), \tag{4.6}$$

where we take $\gamma_l = \beta_l / \|u_0\|_{L^2(\Omega)}$.

Let us calculate the norms of $u^{(N)}$ and $u_t^{(N)}$ in $L^2(0,T;L^2(\Omega))$:

$$\begin{aligned}
||u^{(N)}||^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \int_\Omega \sum_{k=1}^N |w_k^{(N)}(t)|^2 v_k^2(x)\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \sum_{k=1}^N |w_k^{(N)}(t)|^2\mathrm{d}t = \int_0^T \sum_{k=1}^N \Big|\frac{\beta_k\ E(t)}{\|u_0\|_{L^2(\Omega)}}\Big|^2\mathrm{d}t \\
&= \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^N \beta_k^2 \int_0^T E^2(t)\mathrm{d}t
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
||u_t^{(N)}||^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \int_\Omega \sum_{k=1}^N |(w_k^{(N)}(t))_t'|^2 v_k^2(x)\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \sum_{k=1}^N |(w_k^{(N)}(t))_t'|^2\mathrm{d}t = \int_0^T \sum_{k=1}^N \Big|\frac{\beta_k\ E'(t)}{\|u_0\|_{L^2(\Omega)}}\Big|^2\mathrm{d}t \\
&= \frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^N \beta_k^2 \int_0^T |E'(t)|^2\mathrm{d}t.
\end{aligned} \tag{4.8}$$

Next, we get the estimate for $q_l^{(N)}$ which leads to the estimate of the unknown function $f^{(N)}$. Notice that from $(4.3)_1$ and (4.6) we have

$$q_l^{(N)}(t) = \frac{\beta_l}{\|u_0\|_{L^2(\Omega)}} \Big(\lambda_l\, E(t) + E'(t)\Big), \quad \forall l = 1,...,N. \tag{4.9}$$

Let us square both sides of (4.9) and then divide by $1+\lambda_k^2$:

$$\begin{aligned}
\frac{|q_k^{(N)}(t)|^2}{1+\lambda_k^2} &= \frac{\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}} \frac{\Big(\lambda_k E(t) + E'(t)\Big)^2}{1+\lambda_k^2} \le \frac{c\beta_k^2}{\|u_0\|^2_{L^2(\Omega)}}\Big(E^2(t) + \frac{|E'(t)|^2}{1+\lambda_k^2}\Big) \\
&\le \frac{c\beta_k^2}{\|u_0\|_{L^2(\Omega)}}\Big(E^2(t) + |E'(t)|^2\Big).
\end{aligned}$$

Summing up from 1 to N, we derive:

$$\sum_{k=1}^N \frac{|q_k^{(N)}(t)|^2}{1+\lambda_k^2} \le \frac{c}{\|u_0\|^2_{L^2(\Omega)}}\Big(E^2(t) + |E'(t)|^2\Big)\sum_{k=1}^N \beta_k^2, \tag{4.10}$$

i.e., due to Lemma 2 we have

$$\begin{aligned}
||f^{(N)}||^2_{L^2(0,T;W^{-2,2}(\Omega))} &\le \sum_{k=1}^N \frac{|q_k^{(N)}(t)|^2}{1+\lambda_k^2} \\
&\le \frac{c}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^N \beta_k^2 \int_0^T \Big(|E'(t)|^2 + E^2(t)\Big)\mathrm{d}x.
\end{aligned} \tag{4.11}$$

Therefore, estimates (4.7), (4.8) and (4.11) show that sequences $\{u^{(N)}\}$, $\{u_t^{(N)}\}$ are bounded in the space $L^2(0,T;L^2(\Omega))$ and the sequence $\{f^{(N)}\}$ is bounded in the space $L^2(0,T;W^{-2,2}(\Omega))$. Thus, we can choose subsequences $\{u^{(N_j)}\}$, $\{u_t^{(N_j)}\}$ and $\{f^{(N_j)}\}$ weakly converging in the spaces $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;W^{-2,2}(\Omega))$, respectively.

Let us take integral identity $(4.2)_1$ for $N=N_j$:

$$\int_\Omega u_t^{(N_j)}(x,t)\,v_k(x)\mathrm{d}x-\int_\Omega u^{(N_j)}(x,t)\,\Delta v_k(x)\mathrm{d}x=\int_\Omega f^{(N_j)}(x,t)\,v_k(x)\mathrm{d}x. \tag{4.12}$$

We multiply (4.12) by $d_k(t)\in L^2(0,T)$, then sum up from 1 to M, $M\le N_j$ and integrate with respect to t from 0 to T:

$$\int_0^T\int_\Omega u_t^{(N_j)}(x,t)\sum_{k=1}^M v_k(x)\,d_k(t)\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega u^{(N_j)}(x,t)\Delta\Big(\sum_{k=1}^M v_k(x)\,d_k(t)\Big)\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\int_\Omega f^{(N_j)}(x,t)\sum_{k=1}^M v_k(x)\,d_k(t)\mathrm{d}x\mathrm{d}t.$$

Denote $\sum_{k=1}^M v_k(x)\,d_k(t)=\eta(x,t)$. Then, we have

$$\begin{aligned}&\int_0^T\int_\Omega u_t^{(N_j)}(x,t)\,\eta(x,t)\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega u^{(N_j)}(x,t)\,\Delta\eta(x,t)\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_\Omega f^{(N_j)}(x,t)\,\eta(x,t)\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{4.13}$$

where $\eta(x,t)\in L^2(0,T;W^{2,2}(\Omega)\cap\mathring{W}^{1,2}(\Omega))$. Since $\{u^{(N_j)}\}$ and $\{u_t^{(N_j)}\}$ weakly converge in $L^2(0,T;L^2(\Omega))$, and $\{f^{(N_j)}\}$ weakly converges in $L^2(0,T;W^{-2,2}(\Omega))$, we can pass to a limit as $N_j\to\infty$ in equality (4.13):

$$\begin{aligned}&\int_0^T\int_\Omega u_t(x,t)\,\eta(x,t)\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega u(x,t)\,\Delta\eta(x,t)\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_\Omega f(x,t)\,\eta(x,t)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.14}$$

Note that (4.14) is now proved for $\eta(x,t)=\sum_{k=1}^M v_k(x)\,d_k(t)\in L^2(0,T;W^{2,2}(\Omega)\cap\mathring{W}^{1,2}(\Omega))$, and M is an arbitrary natural number. Since the set of all linear combinations $\sum_{k=1}^M v_k(x)\,d_k(t)$ is dense in the space $L^2(0,T;W^{2,2}(\Omega)\cap\mathring{W}^{1,2}(\Omega))$, for every $\eta(x,t)\in L^2(0,T;W^{2,2}(\Omega)\cap\mathring{W}^{1,2}(\Omega))$ there exists a subsequence $\{\eta_l\}$ such that

$$\|\eta_l-\eta\|_{L^2(0,T;W^{2,2}(\Omega)\cap\mathring{W}^{1,2}(\Omega))}\to 0\quad\text{as } l\to\infty.$$

So, for every η_l the equality (4.14) is valid, i.e.,

$$\begin{aligned}&\int_0^T\int_\Omega u_t(x,t)\,\eta_l(x,t)\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega u(x,t)\,\Delta\eta_l(x,t)\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_\Omega f(x,t)\,\eta_l(x,t)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.15}$$

Then, we can pass to a limits as $l \to \infty$ in (4.15):

$$\int_0^T\int_\Omega u_t(x,t)\,\eta(x,t)\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega u(x,t)\,\Delta\eta(x,t)\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega f(x,t)\,\eta(x,t)\mathrm{d}x\mathrm{d}t,$$

for arbitrary function $\eta(x,t)$ from the space $L^2(0,T;W^{2,2}(\Omega)\cap \mathring{W}^{1,2}(\Omega))$.

Next, we need to prove that

$$\int_\Omega |u^{(N)}(x,t)|^2\mathrm{d}x \to E^2(t).$$

Let us denote

$$\varphi^{(N)}(t) = \|u^{(N)}(\cdot,t)\|_{L^2(\Omega)} = \left(\int_\Omega |u^{(N)}|^2 dx\right)^{1/2}. \tag{4.16}$$

Then, $(\varphi^{(N)}(t))'$ is:

$$\begin{aligned}(\varphi^{(N)}(t))' &= \left(\left(\int_\Omega |u^{(N)}|^2 dx\right)^{1/2}\right)'_t = \frac{\int_\Omega u^{(N)} u_t^{(N)} dx}{\|u^{(N)}(\cdot,t)\|_{L^2(\Omega)}} \\ &\le \frac{\left(\int_\Omega |u^{(N)}|^2 dx\right)^{1/2}\left(\int_\Omega |u_t^{(N)}|^2 dx\right)^{1/2}}{\|u^{(N)}(\cdot,t)\|_{L^2(\Omega)}} = \|u_t^{(N)}\|_{L^2(\Omega)}.\end{aligned} \tag{4.17}$$

So, applying estimates (4.16) and (4.17) we get

$$\int_0^T |\varphi^{(N)}(t)|^2 dt = \int_0^T \|u^{(N)}\|^2_{L^2(\Omega)} dt = \|u^{(N)}\|^2_{L^2(0,T;L^2(\Omega))} \tag{4.18}$$

and

$$\int_0^T |(\varphi^{(N)}(t))'|^2 dt \le \int_0^T \|u_t^{(N)}\|^2_{L^2(\Omega)} dt = \|u_t^{(N)}\|^2_{L^2(0,T;L^2(\Omega))}. \tag{4.19}$$

Since the $L^2(0,T;L^2(\Omega))$ - norms of functions $u^{(N)}$ and $u_t^{(N)}$ are finite (see (4.7) and (4.8)) and the estimates (4.18), (4.19) are valid, we conclude $\varphi(t) \in W^{1,2}(0,T)$. The embedding $W^{1,2}(0,T) \hookrightarrow C([0,T])$ is completely continuous (see [5]). Therefore, from $\varphi^{(N)} \rightharpoonup \varphi$ in $W^{1,2}(0,T)$, follows that $\varphi^{(N)} \to \varphi$ in $C([0,T])$, i.e.,

$$\int_\Omega |u^{(N)}(\cdot,t)|^2 dx \to \int_\Omega |u(\cdot,t)|^2 dx. \tag{4.20}$$

Since $\frac{1}{\|u_0\|^2_{L^2(\Omega)}} \sum_{k=1}^N \beta_k^2 E^2(t) \to E^2(t)$ as $N \to \infty$, we can conclude that $\int_\Omega |u(x,t)|^2\mathrm{d}x = E^2(t)$. Due to the condition $E(0) = \|u_0\|_{L^2(\Omega)}$, we obtain $\int_\Omega |u(x,0)|^2\mathrm{d}x = \int_\Omega |u_0(x)|^2\mathrm{d}x$.

In order to prove $u(x,0) = u_0(x)$ we need to get that

$$\lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L^2(\Omega)} = 0. \tag{4.21}$$

Let us estimate the norm $\|u(\cdot,t)-u_0\|_{L^2(\Omega)}$:

$$\begin{aligned}\|u(\cdot,t)-u_0\|_{L^2(\Omega)} \le \|u(\cdot,t)-u^{(N_l)}(\cdot,t)\|_{L^2(\Omega)} \\ + \|u^{(N_l)}(\cdot,t)-u_0^{(N_l)}\|_{L^2(\Omega)} + \|u_0^{(N_l)}-u_0\|_{L^2(\Omega)}.\end{aligned} \tag{4.22}$$

Let us choose an arbitrary small $\varepsilon>0$. From the weak convergence $u^{(N_l)}$ to u and (4.20) we conclude that $u^{(N_l)}$ converges strongly. So, there exists N_* such that for every $N_l>N_*$

$$\|u(\cdot,t)-u^{(N_l)}(\cdot,t)\|_{L^2(\Omega)} \le \varepsilon/3. \tag{4.23}$$

Since $u_0^{(N_l)}$ is the partial sum of the Fourier series of the initial function u_0, the number N_* can be find such that for every $N_l>N_*$

$$\|u_0^{(N_l)}-u_0\|_{L^2(\Omega)} \le \varepsilon/3. \tag{4.24}$$

Let us fix N_l. Then,

$$\begin{aligned}\|u^{(N_l)}(\cdot,t)-u_0^{(N_l)}\|_{L^2(\Omega)} &= \Big(\int_\Omega \Big|\sum_{k=1}^{N_l} w_k^{(N_l)}(t)\, v_k(x) - \sum_{k=1}^{N_l} \beta_k\, v_k(x)\Big|^2 \mathrm{d}x\Big)^{1/2} \\ &= \Big(\sum_{k=1}^{N_l}(w_k^{(N_l)}(t)-\beta_k)^2\Big)^{1/2} = \Big(\sum_{k=1}^{N_l}(w_k^{(N_l)}(t)-w_k^{(N_l)}(0))^2\Big)^{1/2} \le \varepsilon/3\end{aligned} \tag{4.25}$$

for any $t\le\delta(\varepsilon)$. Here we used the fact that functions $w_k^{(N_l)}$ are continuous. Substituting (4.23), (4.24) and (4.25) into inequality (4.22) we arrive at (4.21). This implies that the obtained solution u satisfies the initial condition $u(x,0)=u_0(x)$.

Remark 3. The constructed solutions of problem (3.1)–(3.2) depend on the Fourier coefficients β_k of the initial function u_0. In the case when $u_0=0$ the function $E(t)$ has to satisfy condition $E(0)=0$. Then instead of the coefficients γ_k we can take in (4.6) arbitrary coefficients α_k such that $\sum_{k=1}^{\infty}\alpha_k=1$. So, there is no uniqueness of the solution if $u_0=0$.

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