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A Vieta–Lucas collocation and non-standard finite difference techniques for solving space-time fractional-order Fisher equation

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Abstract. The purpose of the article is to analyze an accurate numerical technique to solve a space-time fractional-order Fisher equation in the Caputo sense. For this purpose, the spectral collocation technique is used, which is based on the Vieta–Lucas approximation. By using the properties of Vieta–Lucas polynomials, this technique reduces the nonlinear equations into a system of ordinary differential equations (ODEs). The non-standard finite difference (NSFD) method converts this system of ODEs into algebraic equations which have been solved numerically. Moreover, the error estimate is investigated for the proposed method. To show the accuracy and efficiency of the technique, the obtained numerical results are compared with the analytical results and existing results of the particular forms of the considered fractional order models through error analysis. The important feature of this article is the exhibition of variations of the field variable for various values of spatial and temporal fractional order parameters for different particular cases.

Keywords: spectral collocation method; space-time fractional Fisher equation; Vieta–Lucas polynomials; NSFD method.

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1 Introduction

Fractional calculus theory was introduced by Nicholas H. Abel and Joseph Liouville. In the last few decades, many scientists, applied mathematicians, engineers, and researchers have focused on fractional calculus because the fractional calculus-based models are widely used in areas such as engineering, biology, physics, hydrology, fluid mechanics, viscoelasticity, and finance, etc. [6, 12, 20, 26]. Fractional order derivatives are an extended form of integer order derivatives. The differential equations containing fractional order derivatives provide more flexibilities as compared to the differential equations of integrer order. Due to their non-local properties, fractional order derivatives are frequently used in mathematical modelling. There are numerous types of fractional differential operators available viz., Caputo-Fabrizio, Caputo, Hadamard, Riesz, Grünwald–Letnikov and Riemann–Liouville etc. Researchers are paying more attention to finding the exact and numerical solutions of fractional order differential equations (FDEs) due to their increasing applications. Sometimes analytical solutions of FDEs are not possible or very difficult, so researchers move on to numerical solutions. There are numerous proposed methods to solve FDEs in the literature, such as Adomian decomposition method, finite difference method, homotopy perturbation method, and Chebyshev wavelets [8,13,19,30]. For additional information on fractional calculus, see [2,9,10,21].

In 1937, Ronald Aylmer Fisher introduced the Fisher equation [11], which is a nonlinear partial differential equation that describes the wave of the progress of an advantageous gene. According to Fisher's equation, the reaction, advection, and dissipation construction of the gene were preserved. Fisher equations have a wide application in the neutron population of a nuclear reaction, logistic population growth, branching Brownian motion, flame propagation, and autocatalytic chemical reactions. In general, there are only a few cases for which analytical solutions to Fisher equations exist. Thus, the crucial need of numerical methods. Therefore, the primary objective of the present research is to develop a proficient computational method to deal with space-time fractional order nonlinear Fisher equations. In addition to developing the numerical method, the present scientific contribution intends to make sure that the developed method performs better than previously existing numerical techniques that handle with these types of complex models.

In this work, we use a combination of the Vieta–Lucas collocation and non-standard finite difference (NSFD) methods in order to solve a space-time fractional order Fisher equation of the following form

$$D_t^{\alpha}u(x,t) = \rho D_x^{\beta}u(x,t) + \kappa u(x,t)\left[1 - u^{\sigma}(x,t)\right], \quad 0 < x < 1, \;\; t \geq 0, \quad (1.1)$$

subject to the initial condition

$$u(x,0) = g(x), \quad (1.2)$$

and boundary conditions

$$u(0,t) = h_1(t), \qquad u(1,t) = h_2(t), \quad (1.3)$$

where $0 < \alpha < 1$, $1 < \beta < 2$, D_t^{α} and D_x^{β} represent the time and space fractional order derivatives, respectively, $\rho > 0$ is the diffusion coefficient, κ be a non-negative parameter, $\sigma = 1, 2, ...$; $g(x)$, $h_1(t)$ and $h_2(t)$ are known functions. For $\alpha = 1$ and $\beta = 2$, the Equation (1.1) will be called the classical Fisher equation.

The NSFD scheme is used as an alternative method to find the accurate solutions of a wide range of problems associated to physical and biological models. In this method, the solution is approximated in series form by Vieta–Lucas polynomials and the unknowns are viewed as functions of t. After that, the collocation method is used to obtain the required number of algebraic equations to find the unknowns. It is a very efficient technique for handling nonlinear fractional-order PDEs over other methods. By this proposed scheme, many researchers have solved various FDEs [1, 14, 29].

In the literature, it is seen that many researchers have proposed various methods for solution of the Fisher's equation (FE). Wazwaz and Gorguis [30] solved the FE with the help of Adomian decomposition method (ADM). Cherif et al. [8] have used the classical homotopy perturbation method (HPM) to solve the space fractional FE. Baranwal et al. [7] have given an analytic solution of the nonlinear time-fractional reaction-diffusion equations by using the variational iteration method and ADM. Khader and Saad [17] used the Chebyshev spectral collocation method for solving the space fractional FE. Sahin et al. [25] have developed a B-spline Galerkin technique for the approximate solution of FE. Al-Khaled [4] has solved Fisher's reaction-diffusion equation using the Sinc collocation technique. Mesgarani et al. [21] have given an approximation of time-fractional nonlinear Burgers-Fisher equation by using the Chebyshev collocation method. Majeed et al. [18] have solved the time-fractional modified FE with the help of cubic B-spline collocation technique. Al Qurashi et al. [5] have developed a residual power series iterative method to solve the time-fractional FE. Saad et al. [24] used an Atangana-Baleanu fractional derivative with spectral collocation methods to approximate the solution of the fractional Fisher's type equations. Secer and Cinar [27] have solved the time-fractional FE using the Jacobi wavelet collocation method. Mohyud-Din and Noor [23] have used the modified variational iteration method for the numerical solution of FE. Khader and Adel [15] have given numerical and theoretical methods based on the compact finite difference and spectral collocation schemes for the space fractional-order Fisher's equation.

The rest part of the article is given as follows. In Section 2, preliminaries containing the definitions and properties of the Caputo derivative, details of the NSFD method, the basic properties of shifted Vieta–Lucas polynomials are given and an approximation formula of the Caputo derivative of any function in form of shifted Vieta–Lucas polynomials are discussed. Section 3 demonstrates the numerical solution of the space-time fractional Fisher equation with the help of a combination of the spectral collocation method and NSFD method. In Section 4, the error bound of the presented technique are discussed. Section 5 contains two numerical examples and their results which is followed by the conclusion section.

2 Preliminaries

Some fundamental mathematical tools and definitions of the fractional calculus theory are introduced in this section which are required for developing the numerical results.

2.1 Definition of Caputo derivative

For a function $h(z)$, the Caputo fractional derivative of order β is defined as [15, 19]

$$D_z^{\beta} h(z) = \frac{1}{\Gamma(l-\beta)} \int_0^z (z-\zeta)^{(l-\beta-1)} h^{(l)}(\zeta) d\zeta, \quad \beta > 0, \zeta > 0,$$

where $l-1 < \beta < l, \quad l \in \mathbb{N}$.

The operator D_z^{β} satisfies the linearity property

$$D_z^{\beta}(\lambda h(z) + \mu g(z)) = \lambda D_z^{\beta} h(z) + \mu D_z^{\beta} g(z).$$

The Caputo fractional derivative of order β of the function z^{ϑ} is represented as

$$D_z^{\beta} z^{\vartheta} = \begin{cases} 0, & \vartheta \in 0, 1, 2, ..., \lceil \beta \rceil - 1, \\ \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+1-\beta)} z^{\vartheta-\beta}, & \vartheta \in \mathbb{N}, \quad \vartheta \geq \lceil \beta \rceil, \end{cases}$$

where $\lceil \beta \rceil$ represents the ceiling function.

2.2 Non-standard finite difference method

According to this method, the discrete first derivative is defined as [22]

$$\frac{du}{dt} = \frac{u_{r+1} - \psi(k) u_r}{\varphi(k)},$$

where u_r be the approximation of $u(t_r)$ with considering the discretization $t_r = rk$ and $\varphi(k)$ and $\psi(k)$ be the two functions depending on the step-size $k = \Delta t$ with

$$\psi(k) = 1 + o(k) \;\text{ and }\; \varphi(k) = k + o(k^2).$$

The two functions $\varphi(k)$ and $\psi(k)$ which depend on different parameters also appear in given differential equations. Moreover, $\varphi(k)$ is a continuous function which satisfies $0 < \varphi(k) < 1$, $k \to 0$. Some examples of $\varphi(k)$ which also satisfy these conditions viz.,

$$\varphi(k) = k, \quad \varphi(k) = \sinh(k), \quad \varphi(k) = \exp(k) - 1, \quad \text{etc.}$$

There is no determined basis for the appropriate choices of the function $\varphi(k)$. More details of the NSFD method can be found in [22].

2.3 Vieta–Lucas polynomials

Here, Vieta–Lucas polynomials $\mathbb{VL}_m(y)$ of degree $m \in \mathbb{N}_0$ defined on the interval $[-2, 2]$ are given by [16]

$$\mathbb{VL}_m(y) = 2\cos(m\phi), \quad \phi = \cos^{-1}\left(y/2\right), \quad \phi \in [0, \pi].$$

The Vieta–Lucas polynomials $\mathbb{VL}_m(y)$ can be created from the following recurrence relation:

$$\mathbb{VL}_m(y) = y\mathbb{VL}_{m-1}(y) - \mathbb{VL}_{m-2}(y), \quad m = 2, 3, ...,$$

with the initial values

$$\mathbb{VL}_0(y) = 2, \qquad \mathbb{VL}_1(y) = y.$$

The expression for the explicit power series form of the Vieta-Lucas polynomials is given as

$$\mathbb{VL}_m(y) = \sum_{j=0}^{\lceil m/2 \rceil} (-1)^j \frac{m\Gamma(m-j)}{\Gamma(j+1)\Gamma(m-2j+1)} y^{m-2j}, \qquad m = 2, 3, ...,$$

where $\lceil m/2 \rceil$ is the integer part of $m/2$. Also, these polynomials $\mathbb{VL}_m(y)$ are orthogonal in $[-2, 2]$ with respect to the weight function $1/\sqrt{4-y^2}$ as

$$\langle \mathbb{VL}_l(y), \mathbb{VL}_m(y) \rangle = \int_{-2}^{2} \frac{1}{\sqrt{4-y^2}} \mathbb{VL}_l(y)\mathbb{VL}_m(y)dy = \begin{cases} 0, & l \neq m \neq 0, \\ 2\pi, & l = m \neq 0, \\ 4\pi, & l = m = 0. \end{cases}$$

2.4 Shifted Vieta–Lucas polynomials

The shifted Vieta–Lucas polynomials $\mathbb{VL}^*_m(y)$ of degree m in y on $[0, 1]$ are given by

$$\mathbb{VL}^*_m(y) = \mathbb{VL}_m(4y - 2).$$

Also, polynomials $\mathbb{VL}^*_m(y)$ are obtained with the help of the recurrence formula as

$$\mathbb{VL}^*_{m+1}(y) = (4y-2)\mathbb{VL}^*_m(y) - \mathbb{VL}^*_{m-1}(y), \quad m = 1, 2, ...,$$

where the initial values are as $\mathbb{VL}^*_0(y) = 2$, $\mathbb{VL}^*_1(y) = 4y - 2$.

The explicit analytical form of $\mathbb{VL}^*_m(y)$ can be generated by

$$\mathbb{VL}^*_m(y) = 2m \sum_{j=0}^{m} (-1)^j \frac{4^{m-j}\Gamma(2m-j)}{\Gamma(j+1)\Gamma(2m-2j+1)} y^{m-j}, \qquad m = 2, 3, ...$$

The polynomials $\mathbb{VL}^*_m(y)$ are orthogonal with respect to the inner product

$$\langle \mathbb{VL}^*_l(y), \mathbb{VL}^*_m(y) \rangle = \int_{0}^{1} \omega(y)\mathbb{VL}^*_l(y)\mathbb{VL}^*_m(y)dy = \begin{cases} 0, & l \neq m \neq 0, \\ 2\pi, & l = m \neq 0, \\ 4\pi, & l = m = 0, \end{cases}$$

where $\omega(y) = 1/\sqrt{y - y^2}$ is the weight function.

Let the function $u(y) \in L^2[0,1]$, which can be expressed as a power series with shifted Vieta–Lucas polynomials as

$$u(y) = \sum_{j=0}^{\infty} b_j \mathbb{VL}_j^*(y),$$

where the coefficients b_j are unknowns. Generally, the above series is approximated till a finite number of terms $(m+1)$ of shifted Vieta–Lucas polynomials

$$u_m(y) = \sum_{j=0}^{m} b_j \mathbb{VL}_j^*(y), \tag{2.1}$$

where $b_j, j = 0, 1, ..., m$ can be obtained by the following expression

$$b_j = \frac{1}{\delta_j \pi} \int_{-2}^{2} \frac{u\left((y+2)/4\right) \mathbb{VL}_j(y)}{\sqrt{4-y^2}} dy, \quad or \;\; b_j = \frac{1}{\delta_j \pi} \int_{0}^{1} \frac{u(y) \mathbb{VL}_j^*(y)}{\sqrt{y-y^2}} dy,$$

where

$$\delta_j = \begin{cases} 4, & j = 0, \\ 2, & j = 1, 2, ..., m. \end{cases}$$

Theorem 1. *Let $u_m(y)$ be an approximate function in terms of shifted Vieta–Lucas polynomials as given in Equation (2.1). Then, the fractional order derivative $\beta > 0$ of the function $u_m(y)$ is defined by*

$$D^{\beta}[u_m(y)] = \sum_{j=\lceil \beta \rceil}^{m} \sum_{p=0}^{j-\lceil \beta \rceil} b_j \; \eta_{j,p}^{(\beta)} \; y^{j-p-\beta},$$

where

$$\eta_{j,p}^{(\beta)} = (-1)^p \frac{4^{j-p} 2j \Gamma(2j-p) \Gamma(j-p+1)}{\Gamma(p+1)\Gamma(2j-2p+1)\Gamma(j-p+1-\beta)}.$$

Proof. The proof is given in [1]. □

3 Numerical solution

The numerical solution of the space-time fractional-order Fisher equation of the type given in Equation (1.1), under the initial and boundary conditions prescribed in Equations (1.2) and (1.3), is determined by approximating of $u(x,t)$ as

$$u(x,t) \approx u_m(x,t) = \sum_{j=0}^{m} b_j(t) \mathbb{VL}_j^*(x), \tag{3.1}$$

where the coefficients $b_j(t), j = 0, 1, ..., m$ are unknown functions.
With the Theorem 1 and the substitution of Equation (3.1) into Equation (1.1),

we have

$$\sum_{j=0}^{m}\frac{\partial^{\alpha}b_j(t)}{\partial t^{\alpha}}\mathbb{VL}_j^*(x)=\rho\sum_{j=\lceil\beta\rceil}^{m}\sum_{p=0}^{j-\lceil\beta\rceil}b_j(t)\ \eta_{j,p}^{(\beta)}\ x^{j-p-\beta}$$
$$+\kappa\Bigg(\sum_{j=0}^{m}b_j(t)\mathbb{VL}_j^*(x)\Bigg)\Bigg[1-\Bigg(\sum_{j=0}^{m}b_j(t)\mathbb{VL}_j^*(x)\Bigg)^{\sigma}\Bigg]. \tag{3.2}$$

Now, we collocate Equation (3.2) at the roots of $\mathbb{VL}_{m+1-\lceil\beta\rceil}(x)$, which are the collocation points, x_r with $r=1,2,...,(m+1-\lceil\beta\rceil)$, so that a system of first order fractional ordinary differential equations (FODEs) is obtained as

$$\sum_{j=0}^{m}\frac{\partial^{\alpha}b_j(t)}{\partial t^{\alpha}}\mathbb{VL}_j^*(x_r)=\rho\sum_{j=\lceil\beta\rceil}^{m}\sum_{p=0}^{j-\lceil\beta\rceil}b_j(t)\ \eta_{j,p}^{(\beta)}\ x_r^{j-p-\beta}$$
$$+\kappa\Bigg(\sum_{j=0}^{m}b_j(t)\mathbb{VL}_j^*(x_r)\Bigg)\Bigg[1-\Bigg(\sum_{j=0}^{m}b_j(t)\mathbb{VL}_j^*(x_r)\Bigg)^{\sigma}\Bigg]. \tag{3.3}$$

Based on Equation (3.1), the boundary conditions (1.3) are written as

$$\sum_{j=0}^{m}\mathbb{VL}_j^*(0)b_j(t)=h_1(t),\quad \sum_{j=0}^{m}\mathbb{VL}_j^*(1)b_j(t)=h_2(t). \tag{3.4}$$

The non-standard finite difference method to find the unknowns $b_j(t)$ for $j=0,1,...,m$ is used to solve the system of the above FODEs. First, we discretize the time-fractional derivative. The given time domain is divided into N equal parts as $t_n=n\Delta t$, $n=0,1,...,N$. By denoting the values of $b_j(t)$ at the point $t=t_n$ as b_j^n and by using the definition of the Caputo derivative, we have

$$\begin{aligned}\frac{\partial^{\alpha}b_j(t_n)}{\partial t^{\alpha}}&=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}(t_n-\zeta)^{-\alpha}\frac{db_j(\zeta)}{d\zeta}d\zeta\\&=\frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{n-1}\frac{b_j^{s+1}-b_j^{s}}{\varphi(k)}\int_{t_s}^{t_{s+1}}(t_n-\zeta)^{-\alpha}d\zeta\\&=\frac{1}{\Gamma(2-\alpha)}\sum_{s=0}^{n-1}\frac{b_j^{s+1}-b_j^{s}}{\varphi(k)}\Big[(t_n-t_s)^{1-\alpha}-(t_n-t_{s+1})^{1-\alpha}\Big].\end{aligned} \tag{3.5}$$

At point $t=t_n$, the systems of FODEs given in Equation (3.3) and Equation (3.4) with the aid of Equation (3.5) can be written in the systems of nonlinear algebraic equations as

$$\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{m}\sum_{s=0}^{n-1}\frac{b_j^{s+1}-b_j^{s}}{\varphi(k)}\Big[(t_n-t_s)^{1-\alpha}-(t_n-t_{s+1})^{1-\alpha}\Big]\mathbb{VL}_j^*(x_r)$$
$$=\rho\sum_{j=\lceil\beta\rceil}^{m}\sum_{p=0}^{j-\lceil\beta\rceil}b_j^n\ \eta_{j,p}^{(\beta)}\ x_r^{j-p-\beta}+\kappa\Bigg(\sum_{j=0}^{m}b_j^n\mathbb{VL}_j^*(x_r)\Bigg)\Bigg[1-\Bigg(\sum_{j=0}^{m}b_j^n\mathbb{VL}_j^*(x_r)\Bigg)^{\sigma}\Bigg] \tag{3.6}$$

and

$$\sum_{j=0}^{m} \mathbb{VL}_j^*(0) b_j^n = h_1(t_n), \quad \sum_{j=0}^{m} \mathbb{VL}_j^*(1) b_j^n = h_2(t_n). \tag{3.7}$$

Now, applying the initial condition at the collocation points x_r, $r = 1, 2, ...,$ $(m+1 \ -\lceil\beta\rceil)$, we obtain a system of algebraic equations as

$$\sum_{j=0}^{m} \mathbb{VL}_j^*(x_r) b_j^0 = g(x_r).$$

After solving this system of equations, we get the value of the initial approximation. With the help of this initial approximation, we can solve the systems of algebraic equations (3.6) and (3.7), and we can obtain the values of unknowns $b_j(t)$, $j = 0, 1, ..., m$.

Algorithm 1.

1. Expand and approximate the function $u(x,t)$ as a linear combination of the first $(m+1)$ terms of $\mathbb{VL}_j^*(x)$.
2. Substituting the approximated value into the considered model.
3. Collocate the obtained equation at points x_r, $r = 1, 2, ..., \big(m+1-\lceil\beta\rceil\big)$, to obtain the first order system of FODEs.
4. Rewrite the boundary conditions in terms of $\mathbb{VL}_j^*(x)$.
5. Applying the non-standard finite difference method to solve the FODEs for the unknown functions $b_j(t)$, $j = 0, 1, ..., m$, results in a system of non-linear algebraic equations that can be solved using an iterative method.
6. Once the unknown functions are determined, the approximate solution of the considered model is computed.

4 Error estimate

Theorem 2. *Let the function $u(x,t) \in [0,1] \times [0,1]$ have continuous partial derivatives up to $(m+1)^{th}$ times and $u_m(x,t)$ be the best approximation of the function $u(x,t)$ defined in Equation (3.1), then we have*

$$\|u(x,t) - u_m(x,t)\| \leq \frac{SCHW}{(m+1)!}(m+2)\pi,$$

where $S = \max[s_k, 0 \leq k \leq m+1]$, $C = \max[\binom{m+1}{k}, 0 \leq k \leq m+1]$, $H = \max[h^{m+1-k}, 0 \leq k \leq m+1]$ and $W = \max[w^k, 0 \leq k \leq m+1]$.

Proof. The Taylor series approximation of the function $u(x,t)$ in the neighborhood of a point (x_0,t_0) is given by

$$\begin{aligned} u(x,t) =& u(x_0,t_0) + \frac{1}{1!}\sum_{k=0}^{1}\binom{1}{k}(x-x_0)^{1-k}(t-t_0)^k u^1_{x^{1-k}t^k}(x_0,t_0) + \dots \\ &+ \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(x-x_0)^{m-k}(t-t_0)^k u^m_{x^{m-k}t^k}(x_0,t_0) \\ &+ \frac{1}{(m+1)!}\sum_{k=0}^{m+1}\binom{m+1}{k}(x-x_0)^{m+1-k}(t-t_0)^k u^{m+1}_{x^{m+1-k}t^k}(\xi,\xi'), \end{aligned}$$

where $x_0 \in [0,1]$, $t_0 \in [0,1]$, $\xi \in (x_0,x)$ and $\xi' \in (t_0,t)$. Assume

$$\begin{aligned} \tilde{u}_m(x,t) =& u(x_0,t_0) + \frac{1}{1!}\sum_{k=0}^{1}\binom{1}{k}(x-x_0)^{1-k}(t-t_0)^k u^1_{x^{1-k}t^k}(x_0,t_0) + \dots \\ &+ \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(x-x_0)^{m-k}(t-t_0)^k u^m_{x^{m-k}t^k}(x_0,t_0), \end{aligned}$$

then,

$$|u(x,t)-\tilde{u}_m(x,t)| = \Big|\frac{1}{(m+1)!}\sum_{k=0}^{m+1}\binom{m+1}{k}(x-x_0)^{m+1-k}(t-t_0)^k u^{m+1}_{x^{m+1-k}t^k}(\xi,\xi')|.$$

Since $u_m(x,t)$ be the best approximation of $u(x,t)$, then,

$$\begin{aligned} &\|u(x,t)-u_m(x,t)\|^2 \le \|u(x,t)-\tilde{u}_m(x,t)\|^2 \le \int_0^1\int_0^1 \omega(x)\omega(t)|u(x,t) \\ &- \tilde{u}_m(x,t)|^2 dxdt \le \int_0^1\int_0^1 \omega(x)\omega(t)|\frac{1}{(m+1)!}\sum_{k=0}^{m+1}\binom{m+1}{k} \\ &\times (x-x_0)^{m+1-k}(t-t_0)^k u^{m+1}_{x^{m+1-k}t^k}(\xi,\xi')|^2 dxdt. \end{aligned}$$

It is assumed that $u(x,t)$ have continuous partial derivatives up to $(m+1)^{th}$ times, therefore, there exist constants $s_0, s_1, \dots, s_{m+1}$ such that

$$\max_{0\le x,t\le 1} u^{m+1}_{x^{m+1-k}t^k}(x,t) \le s_k, 0 \le k \le m+1.$$

Now, we have

$$\begin{aligned} \|u(x,t)-u_m(x,t)\|^2 \le \int_0^1\int_0^1 \omega(x)\omega(t)|\frac{1}{(m+1)!}\sum_{k=0}^{m+1}\binom{m+1}{k} \\ \times(x-x_0)^{m+1-k}(t-t_0)^k s_k|^2 dxdt. \end{aligned} \tag{4.1}$$

Considering $S = \max[s_k, 0 \le k \le m+1]$ and $C = \max[\binom{m+1}{k}, 0 \le k \le m+1]$, then, Equation (4.1) becomes

$$\|u(x,t)-u_m(x,t)\|^2 \le \frac{S^2C^2}{[(m+1)!]^2}\int_0^1\int_0^1 \omega(x)\omega(t)|\sum_{k=0}^{m+1}(x-x_0)^{m+1-k}(t-t_0)^k|^2 dxdt.$$

Now, let $h = \max\{x_0, 1-x_0\}$ and $w = \max\{t_0, 1-t_0\}$, then,

$$\|u(x,t)-u_m(x,t)\|^2 \leq \frac{S^2C^2}{[(m+1)!]^2}\int_0^1\int_0^1 \omega(x)\omega(t)|\sum_{k=0}^{m+1} h^{m+1-k}w^k|^2 dxdt. \quad (4.2)$$

Suppose that $H = \max[h^{m+1-k}, 0 \leq k \leq m+1]$ and $W = \max[w^k, 0 \leq k \leq m+1]$, then, Equation (4.2) can be written as

$$\|u(x,t) - u_m(x,t)\|^2 \leq \frac{S^2C^2H^2W^2}{[(m+1)!]^2}(m+2)^2 \int_0^1\int_0^1 \omega(x)\omega(t)dxdt.$$

Since $\omega(x) = 1/\sqrt{x-x^2}$, so

$$\begin{aligned}\|u(x,t) - u_m(x,t)\|^2 &\leq \frac{S^2C^2H^2W^2}{[(m+1)!]^2}(m+2)^2 \int_0^1\int_0^1 \frac{1}{\sqrt{x-x^2}}\frac{1}{\sqrt{t-t^2}}dxdt,\\ &= \frac{S^2C^2H^2W^2}{[(m+1)!]^2}(m+2)^2\pi^2.\end{aligned}$$

Hence,

$$\|u(x,t) - u_m(x,t)\| \leq \frac{SCHW}{(m+1)!}(m+2)\pi.$$

The proof is completed. □

5 Results and discussions

In this section, we have considered and solved some particular cases of our proposed model and displayed the efficiency and accuracy of the proposed technique through error analyses. We compare the approximate solutions with the exact solutions. Also, we compute the absolute error for them and all numerical computations are done with Wolfram Mathematica version 12.0. The absolute errors in the given tables are $E(x_i,t_j) = |u(x_i,t_j) - u_m(x_i,t_j)|$, where $u(x_i,t_j)$ and $u_m(x_i,t_j)$ are the exact and numerical solution at the points (x_i,t_j).

Example 1. We consider $\rho = 1$, $\kappa = 6$ and $\sigma = 1$. Then the space-time fractional Fisher equation (1.1) is reduced to

$$D_t^\alpha u = D_x^\beta u + 6u(1-u),$$

subject to the initial and boundary conditions as

$$u(x,0) = \frac{1}{(1+e^x)^2}, \quad u(0,t) = \frac{1}{(1+e^{-5t})^2}, \quad u(1,t) = \frac{1}{(1+e^{1-5t})^2}.$$

For $\alpha = 1$ and $\beta = 2$, the problem has the exact solution [3]

$$u(x,t) = (1+e^{x-5t})^{-2}.$$

Tables 1 and 2 show numerical results for different values of x and t of Example 1. These tables display the absolute errors calculated between the exact solutions and approximate solutions. For the purpose of validating the accuracy of this method, we have compared the numerical results obtained with the help of the proposed technique with the results obtained by JWCM [27], HPTM [28], MVIA-II [3], MVIM [23] and ADM [23]. Based on Tables 1 and 2, we can conclude that the present technique is more accurate than the existing methods.

Table 1. The results of absolute errors obtained by our proposed technique and JWCM [27], HPTM [28], when $\alpha = 1, \beta = 2, m = 5$ and $\varphi(k) = \frac{1}{6}(e^{6k} - 1)$ for Example 1.

x	t	Our technique	JWCM [27]	HPTM [28]
0.3	0.10	4.93116E-06	1.27283E-04	2.37371E-03
	0.11	5.77670E-06	3.07913E-04	3.25021E-03
	0.12	6.65863E-06	5.39602E-04	4.33558E-03
	0.13	7.57142E-06	8.17796E-04	5.65691E-03
	0.14	8.50993E-06	1.13750E-03	7.24213E-03
	0.80	2.13666E-05	-	-
0.4	0.10	8.49603E-06	4.86729E-04	1.00792E-03
	0.11	9.20479E-06	3.88085E-04	1.43561E-03
	0.12	9.90480E-06	2.27596E-04	1.98494E-03
	0.13	1.05963E-06	1.00940E-05	2.67612E-03
	0.14	1.12812E-06	2.59033E-04	3.53044E-03
	0.80	2.52914E-06	-	-
0.5	0.10	9.86990E-06	9.44257E-03	2.34485E-04
	0.11	1.04794E-05	9.94952E-03	2.18475E-04
	0.12	1.10593E-05	1.05241E-02	1.62116E-04
	0.13	1.16130E-05	1.11617E-02	5.19537E-05
	0.14	1.21464E-05	1.18571E-02	1.26839E-04
	0.80	2.72284E-05	-	-

Table 2. The results of absolute errors obtained by our proposed technique and other methods, when $\alpha = 1, \beta = 2, m = 5$, and $\varphi(k) = \frac{1}{6}(e^{6k} - 1)$ for Example 1.

x	t	Our technique	MVIA-II [3]	MVIM [23]	ADM [23]
0	0.2	1.11022E-16	7.86375E-04	2.6996E-03	7.2200E-03
0.2		1.18527E-05	3.62574E-04	2.1557E-03	9.8905E-03
0.4		1.55009E-05	4.74527E-05	1.1361E-03	1.0977E-02
0.6		1.42845E-05	2.79875E-04	7.3830E-04	1.0404E-02
0.8		8.72277E-06	3.01061E-04	5.7310E-04	8.5073E-03
1.0		1.11022E-16	1.98852E-04	9.0773E-04	5.8722E-03
0	0.4	2.22045E-16	3.95213E-02	5.0184E-02	5.7530E-02
0.2		2.63392E-05	3.00569E-02	5.2713E-02	1.6115E-01
0.4		3.72095E-05	1.14677E-02	4.1207E-02	1.3911E-01
0.6		3.82192E-05	6.89281E-03	2.2546E-02	1.5158E-01
0.8		2.85813E-05	1.73836E-02	5.2869E-03	1.4353E-01
1.0		0	1.80526E-02	4.2367E-03	1.1933E-01
0	0.8	2.22055E-16	-	-	-
0.2		1.57227E-05	-	-	-
0.4		2.52914E-05	-	-	-
0.6		2.69346E-05	-	-	-
0.8		1.89757E-05	-	-	-
1.0		4.44089E-16	-	-	-

To show how the fractional derivative influences the approximate solution, we have plotted the approximate solution in Figure 1 at $t = 1$ and $m = 4$ for different values of the fractional derivative α and keeping β fixed and in Figure 2 for various values of the fractional derivative β keeping α fixed. From Figures 1 and 2, we can ensure that the approximate solution approaches the exact solution as the values of α and β increase. Figure 5 exhibits the absolute error between the exact and approximate solutions at $t = 0.8$ for $m = 5$ with $\alpha = 1, \beta = 2$.

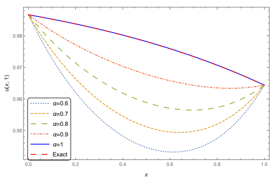

Figure 1. Plots of $u(x, t)$ at $t = 1$ for different values of fractional order parameter α for $m = 4$ and $\beta = 2$ for Example 1.

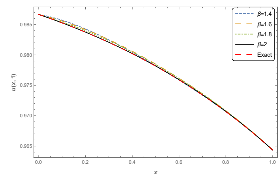

Figure 2. Plots of $u(x, t)$ at $t = 1$ for different values of fractional order parameter β for $m = 4$ and $\alpha = 1$ for Example 1.

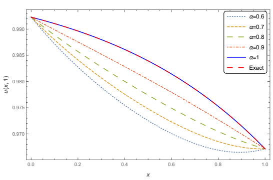

Figure 3. Plots of $u(x, t)$ at $t = 1$ for different values of fractional order parameter α for $m = 4$ and $\beta = 2$ for Example 2.

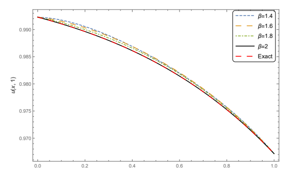

Figure 4. Plots of $u(x, t)$ at $t = 1$ for different values of fractional order parameter β for $m = 4$ and $\alpha = 1$ for Example 2.

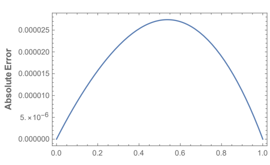

Figure 5. The absolute error at $t = 0.8$ with $\alpha = 1$, $\beta = 2$ and $m = 5$ for Example 1.

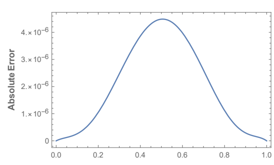

Figure 6. The absolute error at $t = 0.8$ with $\alpha = 1$, $\beta = 2$ and $m = 5$ for Example 2.

Example 2. Consider the prescribed model (1.1) for $\rho = 1, \kappa = 1$ and $\sigma = 6$ so that it is reduced to

$$D_t^{\alpha} u = D_x^{\beta} u + u(1 - u^6),$$

under the initial and boundary conditions as

$$u(x,0)=\left[\frac{1}{2}\tanh\left(\frac{-3}{4}x\right)+\frac{1}{2}\right]^{\frac{1}{3}},u(0,t)=\left[\frac{1}{2}\tanh\left(\frac{15}{8}t\right)+\frac{1}{2}\right]^{\frac{1}{3}},$$

$$u(1,t)=\left[\frac{1}{2}\tanh\left(\frac{-3}{4}+\frac{15}{8}t\right)+\frac{1}{2}\right]^{\frac{1}{3}}.$$

For $\alpha=1$ and $\beta=2$, the problem has the exact solution as

$$u(x,t)=\left[\frac{1}{2}\tanh\left(\frac{-3}{4}\left(x-\frac{5}{2}t\right)\right)+\frac{1}{2}\right]^{\frac{1}{3}}.$$

Table 3 shows the numerical results for different values of x and t of Example 2. In this table, we have calculated the absolute error between the approximate solution and exact solution. For the purpose of validating the accuracy of this method, we have compared the results obtained by the present technique to the results of the MVIM [23] and ADM [23]. Based on Table 3, we can conclude that the proposed technique exhibits better accuracy as compared to other methods. To show how the fractional derivative influences the approximate solution, we have plotted the approximate solutions in Figure 3 for different values of the time fractional derivative α at $\beta=2$ and in Figure 4 for various values of the space fractional derivative β at $\alpha=1$. From Figures 3 and 4, we can ensure that the approximate solutions approach to the exact solutions even for small values of m at $t=1$ as when the values of α and β are increased. Figure 6 displays the absolute error result at $t=0.8$ for $m=5$ with $\alpha=1,\beta=2$.

Table 3. The results of absolute errors obtained by our proposed technique and other methods, when $\alpha=1,\beta=2,m=5$, and $\varphi(k)=1-e^{-k}$ for Example 2.

x	t	Our technique	MVIM [23]	ADM [23]
0		2.22045E-16	4.5414E-02	5.2493E-02
0.2		2.41328E-06	4.1746E-02	7.7955E-02
0.4	0.2	4.40634E-06	3.2328E-02	1.1081E-01
0.6		5.18761E-06	1.9194E-02	1.5138E-01
0.8		3.87627E-07	5.0382E-03	1.9960E-01
1.0		3.33067E-16	7.8583E-03	2.5514E-01
0		8.88178E-16	1.9747E-01	1.2184E-01
0.2		1.39903E-06	8.3997E-02	2.1749E-01
0.4	0.4	2.16603E-06	9.2223E-04	3.4171E-01
0.6		2.11974E-06	4.1063E-02	4.9435E-01
0.8		2.21507E-06	4.1063E-02	6.7402E-01
1.0		1.11022E-16	1.4663E-02	8.7889E-01
0		2.22035E-16	-	-
0.2		2.86876E-06	-	-
0.4	0.8	6.78835E-06	-	-
0.6		7.15564E-06	-	-
0.8		3.43674E-06	-	-
1.0		3.33067E-16	-	-

6 Conclusions

In this article, an approximate solution of the space-time fractional-order Fisher equation is obtained by using the Vieta–Lucas collocation method and the NSFD technique. We derived a formula for the error bound of the proposed numerical method. In order to show the applicability and validity of the suggested technique, the numerical results which are obtained by the applied technique are compared with the results of earlier published works. Based on numerical results, we can conclude that the proposed technique exhibits better efficiency and accuracy as compared to the existing methods. The author is optimist that in the future, the proposed efficient technique can be applied to solve more complex physical models like two-dimensional nonlinear variable order advection-reaction-diffusion equation and 1-D and 2-D nonlinear coupled systems of equations in integer as well as fractional order.

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