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A Discrete Version of the Mishou Theorem Related to Periodic Zeta-Functions

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Abstract. In the paper, we consider simultaneous approximation of a pair of analytic functions by discrete shifts $\zeta_{u_N}(s+ikh_1;\mathfrak{a})$ and $\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})$ of the absolutely convergent Dirichlet series connected to the periodic zeta-function with multiplicative sequence $\mathfrak{a}$, and the periodic Hurwitz zeta-function, respectively. We suppose that $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$, and the set $\{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m+\alpha) : m \in \mathbb{N}_0), 2\pi\}$ is linearly independent over $\mathbb{Q}$.

Keywords: Mishou theorem, periodic zeta-function, periodic Hurwitz zeta-function, universality.

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1 Introduction

Let $s=\sigma+it$ be a complex variable, and $0<\alpha\leqslant 1$ a fixed parameter. The Riemann zeta-function $\zeta(s)$ and Hurwitz zeta-function $\zeta(s,\alpha)$ are defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s} \qquad \text{and} \qquad \zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},$$

and have analytic continuations to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. These functions play an important role in pure mathematics, and have various applications in other natural sciences. One of common feature of the functions $\zeta(s)$ and $\zeta(s,\alpha)$ (for some classes of parameter α) is their universality. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, $H(K)$, $K \in \mathcal{K}$, class of continuous functions on K and analytic in the interior of K, and $H_0(K)$ the subclass of $H(K)$ of non-vanishing on K functions. Then, it is known [1, 18, 20, 29, 39] that there are infinitely many shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, approximating every function $f(s) \in H_0(K)$. Similarly, the set of shifts $\zeta(s + i\tau, \alpha)$ with rational or transcendental α approximating a given function $f(s) \in H(K)$ also is infinite [1,27]. Discrete shifts $\zeta(s+ikh)$ and $\zeta(s+ikh,\alpha)$ with fixed $h > 0$ and $k \in \mathbb{N}$ have an analogical approximation property [1,15,16,19,33,37]. The case of algebraic irrational α is more complicated, was discussed in [2], and the best results were obtained in [38].

H. Mishou in [35] obtained a joint universality theorem for $\zeta(s)$ and $\zeta(s,\alpha)$ with transcendental α. Denote by meas$A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, the Mishou theorem is the following statement.

Theorem 1. *Suppose that the parameter α is transcendental, $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

The problem of algebraic parameter α was discussed in [17].

A discrete analogue of Theorem 1 was proved in [6]. Denote by $\#A$ the cardinality of a set $A \subset \mathbb{R}$, and define the set

$$L(\mathbb{P},\alpha,h,\pi) = \{(\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0), 2\pi/h\},$$

where $\mathbb{P}$ and $\mathbb{N}_0$ are the sets of all prime and non-negative integers, respectively. Then the main result of [6] is

Theorem 2. *Suppose that the set $L(\mathbb{P},\alpha,h,\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K)$, $f_2(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K_1} |\zeta(s+ikh) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s\in K_2} |\zeta(s+ikh,\alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

Generalizations of Theorem 2, including a weighted version, were given in [7, 14] and [34].

The periodic and periodic Hurwitz zeta-functions are generalizations of the Riemann and Hurwitz zeta-functions, respectively. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$

and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0\}$ be two periodic sequences of complex numbers with minimal periods $q_1 \in \mathbb{N}$ and $q_2 \in \mathbb{N}$, respectively. The periodic zeta-function $\zeta(s;\mathfrak{a})$ and periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$, $0 < \alpha \leqslant 1$, are defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \qquad \text{and} \qquad \zeta(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.$$

The periodicity of the sequences $\mathfrak{a}$ and $\mathfrak{b}$ implies, for $\sigma > 1$, the equalities

$$\zeta(s;\mathfrak{a}) = \frac{1}{q_1^s} \sum_{l=1}^{q_1} a_l \zeta\left(s, \frac{l}{q_1}\right) \qquad \text{and} \qquad \zeta(s,\alpha;\mathfrak{b}) = \frac{1}{q_2^s} \sum_{l=0}^{q_2-1} b_l \zeta\left(s, \frac{l+\alpha}{q_2}\right),$$

which give the meromorphic continuations for the functions $\zeta(s;\mathfrak{a})$ and $\zeta,\alpha;\mathfrak{b})$ to the whole complex plane, and

$$\operatorname*{Res}_{s=1} \zeta(s;\mathfrak{a}) = \frac{1}{q_1} \sum_{l=1}^{q_1} a_l \qquad \text{and} \qquad \operatorname*{Res}_{s=1} \zeta(s,\alpha;\mathfrak{b}) = \frac{1}{q_2} \sum_{l=0}^{q_2-1} b_l.$$

We recall that the sequence $\mathfrak{a}$ is multiplicative if $a_1 = 1$ and $a_{mn} = a_m a_n$ for all $(m,n) = 1$. The case of a multiplicative sequence was treated in [31]. Discrete universality for $\zeta(s;\mathfrak{a})$ can be found in [3,13]. Universality of $\zeta(s,\alpha;\mathfrak{b})$ with various types of the parameter α was considered in [8,11,28]. A version of the Mishou theorem for periodic zeta-functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ was obtained in [12].

Theorem 3. [12]. *Suppose that α is transcendental number, and the sequence $\mathfrak{a}$ is multiplicative. Let K_1, K_2 and $f_1(s)$, $f_2(s)$ be the same as in Theorem 1. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon \right\} > 0.$$

The discrete version of Theorem 3 was presented in [13]. Define the set

$$L(\mathbb{P};\alpha,h_1,h_2,\pi) = \{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m+\alpha) : m \in \mathbb{N}_0), 2\pi\},$$

where h_1 and h_2 are positive numbers.

Theorem 4. [13]. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ is linearly independent over $\mathbb{Q}$. Let K_1, K_2 and $f_1(s)$, $f_2(s)$ be the same as in Theorem 1. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K_1} |\zeta(s+ikh;\mathfrak{a}) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s\in K_2} |\zeta(s+ikh,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon \right\} > 0.$$

The aim of this paper, is an extension of Theorem 4 for certain absolutely convergent Dirichlet series related to the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$.

Let $\theta > 1/2$ be a fixed number. For $u > 0$, set

$$v_u(m) = \exp\left\{-(m/u)^\theta\right\}, \quad m \in \mathbb{N},$$
$$v_u(m,\alpha) = \exp\left\{-((m+\alpha)/u)^\theta\right\}, \quad m \in \mathbb{N}_0,$$

where $\exp\{a\} = \mathrm{e}^a$. Define the series

$$\zeta_u(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_u(m)}{m^s} \qquad \text{and} \qquad \zeta_u(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m v_u(m,\alpha)}{(m+\alpha)^s}.$$

Since $v_u(m)$ and $v_u(m,\alpha)$ are exponentially decreasing with respect to m, and a_m and b_m are bounded, the latter series are absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0.

The first universality results with certain $u_T \to \infty$ for $\zeta_{u_T}(s;\{1\})$ were obtained in [21], and discrete version in [32]. The case in short intervals was treated in [23]. A generalization of the above results for $\zeta_{u_T}(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ was presented in [9] and [10]. Similar problems for $\zeta_{u_T}(s,\alpha;\{1\})$ and $\zeta_{u_T}(s,\alpha;\mathfrak{b})$ were discussed in [26] and [5]. The papers [22] and [24] are devoted to extension of Mishou's theorem for absolutely convergent Dirichlet series. In [25], the case of Dirichlet series connected to zeta-functions of certain cusp forms was considered. We also mention the work [30] devoted to the universality of absolutely convergent Dirichlet series with generalized shifts.

We recall a theorem from [4] which extends the Mishou theorem for $\zeta_{u_T}(s;\mathfrak{a})$ and $\zeta_{u_T}(s,\alpha;\mathfrak{b})$ with $u_T \to \infty$. For its statement, we need some notation and definitions. Denote $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define the sets

$$\Omega_1 = \prod_{p\in\mathbb{P}} \gamma_p \qquad \text{and} \qquad \Omega_2 = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. The tori Ω_1 and Ω_2 with the product topology and operation of pointwise multiplication are compact topological Abelian groups. Hence, $\Omega = \Omega_1 \times \Omega_2$ also is a compact topological group, therefore, on $(\Omega, \mathcal{B}(\Omega))$ ($\mathcal{B}(\mathbb{X})$ is the Borel σ-field of the space $\mathbb{X}$), the probability Haar measure m_H exists, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega = (\omega_1, \omega_2)$, $\omega_1 = (\omega_1(p) : p \in \mathbb{P})$, $\omega_2 = (\omega_2(m) : m \in \mathbb{N}_0)$, the elements of Ω, and extend the elements $\omega_1(p)$ to the set $\mathbb{N}$ by the formula

$$\omega_1(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_1^l(p), \quad m \in \mathbb{N}.$$

Let $H(D)$ stand for the space of analytic on D functions endowed with topology of uniform convergence on compacta. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)$-valued random element

$$\underline{\zeta}(s,\alpha,\omega_1,\omega_2;\mathfrak{a},\mathfrak{b}) = (\zeta(s,\omega_1;\mathfrak{a}), \zeta(s,\alpha,\omega_2;\mathfrak{b})),$$

where

$$\zeta(s,\omega_1;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m\omega_1(m)}{m^s} \qquad \text{and} \qquad \zeta(s,\alpha,\omega_2;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m\omega_2(m)}{(m+\alpha)^s}.$$

The main result of [4] is the following theorem.

Theorem 5. [4]. *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative,* α *is transcendental, and* $u_T \to \infty$ *and* $u_T \ll T^2$ *as* $T \to \infty$. *Let* $K_1, K_2 \in \mathcal{K}$ *and* $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. *Then the limit*

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau\in[0,T]: \sup_{s\in K_1}|\zeta_{u_T}(s+i\tau;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta_{u_T}(s+i\tau,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}$$
$$= m_H\Big\{(\omega_1,\omega_2)\in\Omega: \sup_{s\in K_1}|\zeta(s,\omega_1;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta(s,\alpha,\omega_2;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}$$

exists and is positive for all but at most countably many $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$.

Here, and in what follows, the notation $x \ll_\theta y$, $y > 0$, means that there exists a constant $c = c(\theta) > 0$ such that $|x| \leqslant cy$.

We extend Theorem 5 to the discrete case by using the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$.

Theorem 6. *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative, the set* $L(\mathbb{P};\alpha, h_1,h_2,\pi)$ *is linearly independent over* $\mathbb{Q}$, *and* $u_N \to \infty$ *and* $u_N \ll N^2$ *as* $N \to \infty$. *Let* $K_1, K_2 \in \mathcal{K}$ *and* $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. *Then, the limit*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N: \sup_{s\in K_1}|\zeta_{u_N}(s+ikh_1;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}$$
$$= m_H\Big\{(\omega_1,\omega_2)\in\Omega: \sup_{s\in K_1}|\zeta(s,\omega_1;\mathfrak{a})-f_1(s)|<\varepsilon_1,$$
$$\sup_{s\in K_2}|\zeta(s,\alpha,\omega_2;\mathfrak{b})-f_2(s)|<\varepsilon_2\Big\}$$

exists and is positive for all but at most countably many $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$.

We observe that the set $L(\mathbb{P};\alpha,h_1,h_2,\pi)$ is non-empty. We recall that the numbers $\eta_1,\dots,\eta_r$ are algebraically independent over $\mathbb{Q}$ if it does not exist any polynomial $p(s_,\dots,s_r) \neq 0$ with rational coefficients such that $p(\eta_1,\dots,\eta_r) = 0$. The Nesterenko theorem asserts [36] that the numbers π and e^π are algebraically independent over $\mathbb{Q}$. From the latter theorem, it follows that the set

$L(\mathbb{P}; 1/\pi, h_1, h_2, \pi)$ with rational positive h_1 and h_2 is linearly independent over $\mathbb{Q}$. The Nesterenko theorem implies the transcendence of the numbers π and e^π. Suppose, on the contrary, that the set $L(\mathbb{P}; 1/\pi, h_1, h_2, \pi)$ is not linearly independent over $\mathbb{Q}$. Then there exist integers $k_1, \dots, k_{r_1}, \widehat{k}_1, \dots, \widehat{k}_{r_2}$ and $\widetilde{k}$, not all zeros, such that

$$\begin{aligned} &k_1 h_1 \log p_1 + \ldots + k_{r_1} h_1 \log p_{r_1} + \widehat{k}_1 h_2 \log(m_1 + 1/\pi) + \ldots \\ &\qquad + \widehat{k}_{r_2} h_2 \log(m_{r_2} + 1/\pi) + \widetilde{k}\pi = 0. \end{aligned}$$

Hence,

$$p_1^{l_1} \ldots p_{r_1}^{l_{r_1}} (m_1 + 1/\pi)^{\widehat{l}_1} \ldots (m_{r_2} + 1/\pi)^{\widehat{l}_{r_2}} \mathrm{e}^{\widetilde{l}\pi} = 1$$

with some integers $l_1, \dots, l_{r_1}, \widehat{l}_1, \dots, \widehat{l}_{r_2}$ and $\widetilde{l}$, and this contradicts the algebraic independence of the numbers π and e^π. Similarly, the equalities

$$\begin{aligned} &k_1 h_1 \log p_1 + \ldots + k_{r_1} h_1 \log p_{r_1} + \widehat{k}_1 h_2 \log(m_1 + 1/\pi) + \ldots \\ &\qquad + \widehat{k}_{r_2} h_2 \log(m_{r_2} + 1/\pi) = 0, \\ &k_1 h_1 \log p_1 + \ldots + k_{r_1} h_1 \log p_{r_1} + \widetilde{k}\pi = 0 \end{aligned}$$

contradict the transcendence of the numbers π and e^π, respectively. Moreover, it is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$, therefore, the equality

$$k_1 h_1 \log p_1 + \ldots + k_{r_1} h_1 \log p_{r_1} = 0$$

gives again a contradiction.

A proof of Theorem 6 is probabilistic, it is based on a limit theorem in the space $H^2(D)$ for periodic zeta-functions obtained in [13]. Moreover, the application of a limit theorem requires a certain estimate in the mean for the metric in $H^2(D)$.

2 The main equality

We start with recalling the metric in $H^2(D)$. For $g_1, g_2 \in H(D)$, let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact embedded set such that D is the union of the sets K_l, and each compact set of D lies in some K_l. Then, ρ is a metric which induces the topology of uniform convergence on compacta in the space $H(D)$. For $\underline{g}_l = (g_{l1}, g_{l2})$, $l = 1, 2$, let

$$\rho_2(\underline{g}_1, \underline{g}_2) = \max\left(\rho(g_{11}, g_{12}), \rho(g_{21}, g_{22})\right).$$

Then, ρ_2 is a metric in $H^2(D)$ inducing the product topology.

In this section, we consider the mean value of the distance between $\underline{\zeta}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b})$ and $\underline{\zeta}_{u_N}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b})$, where

$$\underline{\zeta}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b}) = \left(\zeta(s+ikh_1;\mathfrak{a}),\zeta(s+ikh,\alpha;\mathfrak{b})\right),$$
$$\underline{\zeta}_{u_N}(s+ik\underline{h},\alpha;\mathfrak{a},\mathfrak{b}) = \left(\zeta_{u_N}(s+ikh_1;\mathfrak{a}),\zeta_{u_N}(s+ikh,\alpha;\mathfrak{b})\right)$$

and $\underline{h}=(h_1,h_2)$. For this, we apply the following lemmas.

Lemma 1. *Suppose that $u_N\to\infty$ and $u_N\ll N^2$ as $N\to\infty$. Then, for every $h_1>0$,*

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\rho\left(\zeta(s+ikh_1;\mathfrak{a}),\zeta_{u_N}(s+ikh;\mathfrak{a})\right)=0.$$

The lemma is Lemma 1 from [10].

Lemma 2. *For every fixed $\sigma>1/2$, $h_2>0$ and $t\in\mathbb{R}$, the estimate*

$$\sum_{k=0}^{N}\left|\zeta(\sigma+ikh_2+it,\alpha;\mathfrak{b})\right|^2\ll_{\sigma,\alpha,\mathfrak{b}} N(1+|t|)$$

is valid.

A proof of lemma is given in [13].

Lemma 3. *Under hypotheses of Lemma 1,*

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\rho\left(\zeta(s+ikh_2,\alpha;\mathfrak{b}),\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})\right)=0.$$

Proof. In virtue of the definition of the metric ρ, it is sufficient to show that the equality

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\left|\zeta(s+ikh_2,\alpha;\mathfrak{a})-\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})\right|=0$$

is true for every compact set $K\subset D$. Using the Mellin formula

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(z)b^{-z}\,\mathrm{d}z=\mathrm{e}^{-b},\tag{2.1}$$

where $\Gamma(z)$ denotes the Euler gamma-function, and $a,b>0$, leads, for $\sigma>1/2$, to the integral representation

$$\zeta_{u_N}(s,\alpha;\mathfrak{b})=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,\alpha;\mathfrak{b})l_{u_N}(z)\,\mathrm{d}z,\tag{2.2}$$

where θ comes from the definition of $v_{u_N}(m,\alpha)$, and

$$l_{u_N}(z)=\frac{1}{\theta}\Gamma\left(\frac{z}{\theta}\right)u_N^z.$$

Actually, in view of (2.1),

$$\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\frac{1}{(m+\alpha)^z}\frac{1}{\theta}\Gamma\left(\frac{z}{\theta}\right)u_N^z\,\mathrm{d}z=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\Gamma\left(\frac{z}{\theta}\right)\left(\frac{m+\alpha}{u_N}\right)^{(-z/\theta)\theta}\mathrm{d}z$$
$$=\exp\left\{-\left((m+\alpha)/u_N\right)^{\theta}\right\}.$$

Therefore, since $\theta+\sigma>1$ for $\sigma>1/2$, we have

$$\zeta_{u_N}(s,\alpha;\mathfrak{b})=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\sum_{m=0}^{\infty}\frac{b_m v_{u_N}(m,\alpha)}{(m+\alpha)^{s+z}}l_{u_N}(z)\,\mathrm{d}z$$
$$=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,\alpha;\mathfrak{b})l_{u_N}(z)\,\mathrm{d}z.$$

Fix a compact set $K\subset D$. Then, there exists a number $0<\delta<\frac{1}{6}$ such that $1/2+2\delta\leqslant\sigma\leqslant1-\delta$ for $s=\sigma+it\in K$. Let $\theta=1/2+\delta$ and $\theta_1=1/2+\delta-\sigma$. Then, $-1/2+2\delta\leqslant\theta_1\leqslant-\delta$. Therefore, the integrand of (2.2), in the strip $\theta_1\leqslant\sigma\leqslant\theta$, has a simple pole at $z=0$ and a possible simple pole at $z=1-s$. Hence, by the residue theorem, we find, for $s\in K$,

$$\zeta_{u_N}(s,\alpha;\mathfrak{b})-\zeta(s,\alpha;\mathfrak{b})=\frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty}\zeta(s+z,\alpha;\mathfrak{b})l_{u_N}(z)\,\mathrm{d}z+R_N(s,\alpha;\mathfrak{b}),$$

where

$$R_N(s,\alpha;\mathfrak{b})=\begin{cases}0 & \text{if } r\overset{\text{def}}{=}\operatorname*{Res}_{s=1}\zeta(s,\alpha;\mathfrak{b})=0,\\ rl_{u_N}(1-s) & \text{otherwise.}\end{cases}$$

The latter equality, for $s=\sigma+it\in K$, gives

$$\begin{aligned}&\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})-\zeta(s+ikh_2,\alpha;\mathfrak{b})\\&\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\zeta\left(\frac12+\delta+it+ikh_2+i\tau,\alpha;\mathfrak{b}\right)l_{u_N}\left(\frac12+\delta-\sigma+i\tau\right)\mathrm{d}\tau\\&\qquad+R_N(s+ikh_2,\alpha;\mathfrak{b})\\&\quad\ll\int_{-\infty}^{\infty}\left|\zeta\left(\frac12+\delta+ikh_2+i\tau,\alpha;\mathfrak{b}\right)\right|\sup_{s\in K}\left|l_{u_N}\left(\frac12+\delta-s+i\tau\right)\right|\mathrm{d}\tau\\&\qquad+\sup_{s\in K}|R_N(s+ikh_2,\alpha;\mathfrak{b})|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh_2,\alpha;\mathfrak{b})-\zeta_{u_N}(s+ikh_2,\alpha;\mathfrak{b})|\\&\qquad\ll\int_{-\infty}^{\infty}\left(\frac{1}{N+1}\sum_{k=0}^{N}|\zeta\left(1/2+\delta+ikh_2+i\tau,\alpha;\mathfrak{b}\right)|\right)\\&\qquad\quad\times\sup_{s\in K}|l_{u_N}\left(1/2+\delta-s+i\tau\right)|\,\mathrm{d}\tau\\&\qquad\quad+\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|R_N(s+ikh_2,\alpha;\mathfrak{b})|\overset{\text{def}}{=}I_N+S_N.\end{aligned}\tag{2.3}$$

For estimating of the integral I_N, we apply Lemma 2. The Cauchy-Schwarz inequality and Lemma 2 yield

$$\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(1/2+\delta+ikh_2+i\tau,\alpha;\mathfrak{b}\right)\right| \ll \left(\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(1/2+\delta+ikh_2+i\tau,\alpha;\mathfrak{b}\right)\right|^2\right)^{1/2} \ll (1+|\tau|)^{1/2}. \quad (2.4)$$

By the definition of $l_{u_N}(s)$, using the classical bound for the gamma-function

$$\Gamma(\sigma+it) \ll \exp\{-c(1+|t|)\}, \quad c>0, \quad (2.5)$$

which is uniform in σ lying in every fixed interval $[\sigma_1,\sigma_2]$, we find that, for $s\in K$,

$$l_{u_N}\left(1/2+\delta-s+i\tau\right) \ll_\delta u_N^{1/2+\delta-\sigma}\exp\left\{-\frac{c}{\theta}(1+|\tau-t|)\right\} \ll_{\delta,K} u_N^{-\delta}\exp\{-c_1|\tau|\}, \quad c_1>0,$$

because of boundedness of t. This and (2.4) show that

$$I_N \ll_{\delta,h_2,\alpha,\mathfrak{b},K} u_N^{-\delta}\int_{-\infty}^{\infty}(1+|\tau|)^{1/2}\exp\{-c_1|\tau|\}\,\mathrm{d}\tau \ll_{\delta,h_2,\alpha,\mathfrak{b},K} u_N^{-\delta}. \quad (2.6)$$

By the definitions of $R_N(s,\alpha;\mathfrak{b})$ and $l_{u_N}(s)$, and (2.5), for $s\in K$, we have

$$R_N(s+ikh_2,\alpha;\mathfrak{b}) \ll_{\delta,\alpha,\mathfrak{b}} u_N^{1-\sigma}\exp\{-c_2(1+kh_2|t|)\} \ll_{\delta,\alpha,\mathfrak{b},K} u_N^{1/2-2\delta}\exp\{-c_3(1+kh_2)\}, \quad c_2,c_3>0.$$

Therefore,

$$S_N \ll_{\delta,K} u_N^{1/2-2\delta}\frac{1}{N}\sum_{k=0}^{N}\exp\{-c_3(1+kh_2)\} \ll_{\delta,\alpha,\mathfrak{b},K} u_N^{1/2-2\delta} \times\left(\frac{\log N}{N}+\frac{1}{N}\sum_{k\geqslant\log N}\exp\{-c_3kh_2\}\right) \ll_{\delta,\alpha,\mathfrak{b},K,h_2} u_N^{1/2-2\delta}\frac{\log N}{N}.$$

Thus, in view of (2.6),

$$I_N+S_N \ll_{\delta,h_2,\alpha,\mathfrak{b},K} u_N^{-\delta}+u_N^{1/2-2\delta}\frac{\log N}{N}.$$

Since $u_N\to\infty$ and $u_N\ll N^2$, this shows that

$$\lim_{N\to\infty}(I_N+S_N)=0,$$

and, by (2.3), the lemma is proved. □

Now, we state the main result on the closeness of $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$, $\underline{\zeta}_{u_N}(s,\alpha;\mathfrak{a},\mathfrak{b})$.

Lemma 4. *Suppose that $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$. Then, for every positive h_1 and h_2,*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho_2 \left(\underline{\zeta}(s+ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_{u_N}(s+ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}) \right) = 0.$$

Proof. By the definition of the metric ρ_2, it suffices to prove that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho \left(\zeta(s+ikh_1; \mathfrak{a}), \zeta_{u_N}(s+ikh_1; \mathfrak{a}) \right) = 0$$

and

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho \left(\zeta(s+ikh_2, \alpha; \mathfrak{b}), \zeta_{u_N}(s+ikh_2, \alpha; \mathfrak{b}) \right) = 0.$$

Therefore, the lemma is consequence of Lemmas 1 and 3. □

3 Limit theorems

Recall that $H(D)$ is the space of analytic on D functions. The proof of Theorem 6 relies on a discrete limit theorem for $\underline{\zeta}_{u_N}(s, \alpha; \mathfrak{a}, \mathfrak{b})$ in the space $H^2(D)$ on weakly convergent probability measures. For brevity, let $P_{\underline{\zeta}}$ be the distribution of the random element $\underline{\zeta}(s, \alpha, \omega_1, \omega_2; \mathfrak{a}, \mathfrak{b})$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H \left\{ (\omega_1, \omega_2) \in \Omega : \underline{\zeta}(s, \alpha, \omega_1, \omega_2; \mathfrak{a}, \mathfrak{b}) \in A \right\}, \quad A \in \mathcal{B}(H^2(D)).$$

For $A \in \mathcal{B}(H^2(D))$, define

$$P_N(A) = \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \underline{\zeta}(s+ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}) \in A \right\}.$$

Lemma 5. [13]. *Suppose that the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$. Then, P_N converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.*

Lemmas 4 and 5 lead to a limit theorem for $\zeta_{u_N}(s, \alpha; \mathfrak{a}, \mathfrak{b})$. Let, for $A \in \mathcal{B}(H^2(D))$,

$$P_{N,u_N}(A) = \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \underline{\zeta}_{u_N}(s+ikh, \alpha; \mathfrak{a}, \mathfrak{b}) \in A \right\}.$$

Theorem 7. *Suppose that the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$, and $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$. Then, P_{N,u_N} converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.*

Proof. Let ξ_N be a random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mu)$ and having the distribution $\mu\{\xi_N{=}k\}{=}1/(N{+}1)$, $k{=}0, 1, \ldots, N$. We will use the equivalent of weak convergence of probability measures in terms of closed sets, namely, if P and P_n, $n \in \mathbb{N}$, are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, then P_n, as $n \to \infty$, converges weakly to P if and only if

$$\limsup_{n\to\infty} P_n(F) \leqslant P(F)$$

for every closed set $F \subset \mathbb{X}$. Fix a closed set $F \subset H^2(D)$, $\varepsilon > 0$, and define the set

$$F_\varepsilon = \left\{ \underline{g} \in H^2(D) : \inf_{\underline{\widehat{g}} \in F} \left\{ \rho_2 \left(\underline{g}, \underline{\widehat{g}} \right) \leqslant \varepsilon \right\} \right\}.$$

Then, the set F_ε is closed as well. Define two $H^2(D)$-valued random elements

$$\underline{X}_N = \underline{X}_N(s) = \underline{\zeta}(s + i\xi_N \underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}), \quad \underline{Y}_N = \underline{Y}_N(s) = \underline{\zeta}_{u_N}(s + i\xi_N \underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}).$$

By the definition of the random variable ξ_N, the random elements $\underline{X}_N$ and $\underline{Y}_N$ have the distributions P_N and P_{N,u_N}, respectively. Moreover,

$$\{\underline{Y}_N \in F_\varepsilon\} \subset \{\underline{X}_N \in F\} \cup \{\rho_2(\underline{X}_N, \underline{Y}_N) \geqslant \varepsilon\}.$$

Hence,

$$\begin{aligned} &\mu(F_\varepsilon) \leqslant \mu(F) + \mu\{\rho_2(\underline{X}_N, \underline{Y}_N) \geqslant \varepsilon\}, \\ &P_{N,u_N}(F_\varepsilon) \leqslant P_N(F) + \mu\{\rho_2(\underline{X}_N, \underline{Y}_N) \geqslant \varepsilon\}. \end{aligned} \tag{3.1}$$

By Lemma 5 and equivalent of weak convergence in terms of closed sets,

$$\limsup_{N\to\infty} P_N(F) \leqslant P_{\underline{\zeta}}(F). \tag{3.2}$$

Moreover, Lemma 4 implies that

$$\begin{aligned} \limsup_{N\to\infty} \mu\{\rho_2(\underline{X}_N, \underline{Y}_N) \geqslant \varepsilon\} &= \limsup_{N\to\infty} \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \\ &\rho_2 \left(\underline{\zeta}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_{u_N}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}) \right) \geqslant \varepsilon \Big\} \\ &\leqslant \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=0}^{N} \rho_2 \left(\underline{\zeta}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_{u_N}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}) \right) = 0. \end{aligned}$$

Thus, in view of (3.1) and (3.2),

$$\limsup_{N\to\infty} P_{N,u_N}(F_\varepsilon) \leqslant P_{\underline{\zeta}}(F).$$

Letting $\varepsilon \to +0$, we obtain $\limsup_{N\to\infty} P_{N,u_N}(F) \leqslant P_{\underline{\zeta}}(F)$, and this together with equivalent of weak convergence in terms of closed sets proves the theorem. □

Theorem 7 implies the weak convergence for the corresponding probability measures in the space $\mathbb{R}^2$. For $A \in \mathcal{B}(\mathbb{R}^2)$, define

$$\begin{aligned} Q_{N,u_N}(A) = &\frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \left(\sup_{s\in K_1} |\zeta_{u_N}(s + ikh_1; \mathfrak{a}) - f_1(s)|, \right.\right. \\ &\left.\left. \sup_{s\in K_2} |\zeta_{u_N}(s + ikh_2, \alpha; \mathfrak{b}) - f_2(s)| \right) \in A \right\}. \end{aligned}$$

Corollary 1. Suppose that the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$, and $u_N \to \infty$ and $u_N \ll N^2$ as $N \to \infty$. Let K_1, K_2 and $f_1(s)$, $f_2(s)$ be as in Theorem 6. Then Q_{N,u_N} converges weakly to the measure

$$m_H \Big\{ (\omega_1, \omega_2) \in \Omega : \Big(\sup_{s \in K_1} |\zeta(s, \omega_1; \mathfrak{a}) - f_1(s)|, \\ \sup_{s \in K_2} |\zeta(s, \alpha, \omega_2; \mathfrak{b}) - f_2(s)| \Big) \in A \Big\}, \quad A \in \mathcal{B}(\mathbb{R}^2), \tag{3.3}$$

as $N \to \infty$.

Proof. Consider the mapping $u : H^2(D) \to \mathbb{R}^2$ defined by

$$u(g_1, g_2) = \Big(\sup_{s \in K_1} |g_1(s) - f_1(s)|, \sup_{s \in K_2} |g_2(s) - f_2(s)| \Big), \quad g_1, g_2 \in H(D).$$

Then, the mapping u is continuous. Actually, suppose that $(g_{N1}, g_{N2}) \to (g_1, g_2)$ as $N \to \infty$ in the space $H^2(D)$. Since the convergence in $H(D)$ is uniform on compact sets, we have

$$\lim_{N \to \infty} \sup_{s \in K_j} |g_{Nj}(s) - g_j(s)| = 0, \quad j = 1, 2.$$

Therefore, using the triangle inequality, we obtain that

$$\Big(\sup_{s \in K_j} |g_{Nj}(s) - f_j(s)| - \sup_{s \in K_j} |g_j(s) - f_j(s)| \Big) \leqslant \sup_{s \in K_j} |g_{Nj} - g_j(s)| \xrightarrow[N \to \infty]{} 0,$$

for $j = 1, 2$. This proves that

$$\lim_{N \to \infty} u(g_{N1}, g_{N2}) = u(g_1, g_2),$$

i.e., u is continuous.

By the definitions of u, P_{N,u_N} and Q_{N,u_N}, for $A \in \mathcal{B}(\mathbb{R}^2)$, we have

$$Q_{N,u_N}(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \underline{\zeta}_{u_N}(s + ik\underline{h}, \alpha; \mathfrak{a}, \mathfrak{b}) \in u^{-1}A \Big\} \\ = P_{N,u_N}(u^{-1}A) = P_{N,u_N} u^{-1}(A),$$

i.e., $Q_{N,u_N} = P_{N,u_N} u^{-1}$. Therefore, the continuity of u, Theorem 7 and the preservation of weak convergence under continuous mappings, show that Q_{N,u_N} converges weakly to $P_{\underline{\zeta}} u^{-1}$, i.e., to the measure (3.3) as $N \to \infty$. □

4 Proof of Theorem 6

Theorem 6 follows from Corollary 1, weak convergence in $\mathbb{R}^2$, support of the measure $P_{\underline{\zeta}}$, and the Mergelyan theorem on approximation of analytic functions by polynomials. We recall that the support of the measure $P_{\underline{\zeta}}$ is a minimal closed set $S_{\underline{\zeta}}$ such that $P_{\underline{\zeta}}(S_{\underline{\zeta}}) = 1$. The set $S_{\underline{\zeta}}$ consists of all $\underline{g} \in H^2(D)$, for every open neighbourhood G of which the inequality $P_{\underline{\zeta}}(G) > 0$ is true.

Define $S(\mathfrak{a}) = \{g \in H(D) : \text{either } g(s) \neq 0, \text{ or } g(s) \equiv 0\}$ and $S(\mathfrak{b}) = H(D)$.

Lemma 6. [13]. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}}$ is the set $S(\mathfrak{a}) \times S(\mathfrak{b})$.*

The next lemma is a version of the Mergelyan theorem on approximation of analytic functions by polynomials.

Lemma 7. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $g(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(s)$ such that*

$$\sup_{s\in K} |g(s) - p_\varepsilon(s)| < \varepsilon.$$

Proof. (Proof of Theorem 6). It is well known that the weak convergence of probability measures is equivalent to that of the corresponding distribution functions. Recall that the distribution function $D_n(x_1, x_2)$, as $n \to \infty$, converges weakly to the distribution function $D(x_1, x_2)$ if

$$\lim_{n\to\infty} D_n(x_1, x_2) = D(x_1, x_2)$$

for $(x_1, x_2) \in \mathbb{R}^2$ such that x_1 and x_2 are continuity points of the functions $D(x_1, +\infty)$ and $D(+\infty, x_2)$, respectively.

Define the distribution functions

$$\begin{aligned} F_N(\varepsilon_1, \varepsilon_2) =& \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \sup_{s\in K_1} |\zeta_{u_N}(s + ikh_1; \mathfrak{a}) - f_1(s)| < \varepsilon_1, \\ & \sup_{s\in K_2} |\zeta_{u_N}(s + ikh_2, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon_2 \Big\} \\ F(\varepsilon_1, \varepsilon_2) =& m_H \Big\{ (\omega_1, \omega_2) \in \Omega : \sup_{s\in K_1} |\zeta(s, \omega_1; \mathfrak{a}) - f_1(s)| < \varepsilon_1, \\ & \sup_{s\in K_2} |\zeta(s, \alpha, \omega_2; \mathfrak{b}) - f_2(s)| < \varepsilon_2 \Big\}. \end{aligned}$$

Then, by Corollary 1, we have that $F_N(\varepsilon_1, \varepsilon_2)$ converges weakly to $F(\varepsilon_1, \varepsilon_2)$ as $N \to \infty$. Thus,

$$\lim_{N\to\infty} F_N(\varepsilon_1, \varepsilon_2) = F(\varepsilon_1, \varepsilon_2), \tag{4.1}$$

where ε_1 and ε_2 are continuity points of the distribution functions $F(\varepsilon_1, +\infty)$ and $F(+\infty, \varepsilon_2)$, respectively. Since the distribution functions $F(\varepsilon_1, +\infty)$ and $F(+\infty, \varepsilon_2)$ have at most countable sets of discontinuity points, the equality (4.1) is true for all but at most countably many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

It remains to show that $F(\varepsilon_1, \varepsilon_2) > 0$. For this, we will apply Lemma 7.

By Lemma 7, there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1} |f_1(s) - e^{p_1(s)}| < \frac{\varepsilon_1}{2}, \quad \sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon_2}{2}. \tag{4.2}$$

Define the set

$$G_{\varepsilon_1,\varepsilon_2} = \Big\{ (g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - \mathrm{e}^{p_1(s)}| < \frac{\varepsilon_1}{2}, \\ \sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon_2}{2} \Big\}.$$

The point $(\mathrm{e}^{p_1(s)}, p_2(s))$, in view of Lemma 6, is an element of the support of the measure $P_{\underline{\zeta}}$. Therefore,

$$P_{\underline{\zeta}}(G_{\varepsilon_1,\varepsilon_2}) > 0. \tag{4.3}$$

Define one more set

$$\mathcal{G}_{\varepsilon_1,\varepsilon_2} = \Big\{ (g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon_1, \\ \sup_{s\in K_2} |g_2(s) - f_2(s)| < \varepsilon_2 \Big\}.$$

In view of equalities (4.2), we have the inclusion $G_{\varepsilon_1,\varepsilon_2} \subset \mathcal{G}_{\varepsilon_1,\varepsilon_2}$. Therefore, by (4.3),

$$P_{\underline{\zeta}}(\mathcal{G}_{\varepsilon_1,\varepsilon_2}) \geqslant P_{\underline{\zeta}}(G_{\varepsilon_1,\varepsilon_2}) > 0.$$

This and the definitions of $P_{\underline{\zeta}}$ and $\mathcal{G}_{\varepsilon_1,\varepsilon_2}$ gives the positivity of $F(\varepsilon_1, \varepsilon_2)$. The theorem is proved. □

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