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A New Approach for Solving a Nonlinear System of Second-Order BVPs

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Abstract. In this paper, we introduce a new approach based on the Reproducing Kernel Method (RKM) for solving a nonlinear system of second-order Boundary Value Problems (BVPs) without the Gram-Schmidt orthogonalization process. What motivates us to use the RKM without the Gram-Schmidt orthogonalization process is its easy implementation, elimination of the Gram-Schmidt process, fewer calculations, and high accuracy. Finally, the compatibility of numerical results and theorems demonstrates that the Present method is effective.

Keywords: reproducing kernel method, system of second-order boundary value problem, convergence analysis, error analysis.

AMS Subject Classification: 25T35; 35T45; 45T55.

1 Introduction

Some phenomena in nature can be modeled by nonlinear ordinary differential equation systems, as used in physics, engineering, finance, biology, and so on [2, 3, 13]. Many numerical methods, such as Legendre wavelet collocation method, novel Petrov-Galerkin method, RBF (Radial Base Function) collocation approach, (HPM) Homotopy Perturbation Method as an especial case of (HAM) Homotopy Analysis Method, discontinuous Galerkin method, and (VIM) Variational Iteration Method, have been provided to solve linear and nonlinear ordinary differential equation systems [5, 6, 7, 9, 10].

In the last decade, the RKM has been considered as a solution for different types of differential equations and systems of differential equations. For example, the RKM has been used to nonlinear singular boundary value problems, nonlinear C-q-fractional IVPs, linear systems of second-order boundary value problems, systems of linear Volterra integral and equations, solve coupled systems of fractional order [12,15,16,17,20]. The advantages of the RKM are easy implementation and acceptable accuracy in estimating approximate solutions. Recently, the RKM has been used in different forms, with and without the Gram-Schmidt orthogonalization process. The numerical results show that the RKM without the Gram-Schmidt orthogonalization process has greater accuracy [18,19]. In addition, due to the elimination of the Gram-Schmidt orthogonalization process, we have a reduced volume of calculations. Lower calculation volume and higher accuracy encouraged us to present our own method.

In the present method, spaces, bases, and collocation points are effective in determining the accuracy of the method. For this purpose, we used the desirable space as a direct sum of two other spaces according to the type of operator and the order of the derivative. Then, we introduced the bases of these two spaces using the unit vector and the bases of the used space. Next, using the self-adjoint operator and applying some relations, we constructed the coefficient matrix. Finally, to obtain the unknown vector, we multiply the inverse of the coefficient matrix by the vector on the right. Simplicity of implementation, acceptable accuracy, low volume of operation, and the elimination of the Gram-Schmidt process are prominent features of the present method.

In this paper, we compare the present method with a combination of the HPM and RKM (briefly, HP-RKM). The perturbation method and homotopy technique are traditional methods. The HPM does not require the discretization of the problem, making it suitable for finding the approximate solution without discretization of the problem and it is a special case of HAM (Homotopy Analysis Method). The HP-RKM was successfully applied to (BVPs) and partial differential equations [8]. We will show that our method is much more efficient than the HP-RKM. This paper has been structured into the following chapters.

In Section 2, the definitions and essential theorems related to the present method are provided. Additionally, at the end of this section, the present method is implemented for systems of nonlinear equations. In Section 3, theorems related to the existence of a solution, convergence, and error estimation are presented. In Section 4, three examples are solved using the present method and their numerical results are presented in the form of tables and figures. Finally, we conclude in the last section.

Consider the following nonlinear system of BVPs in the reproducing kernel space

$$\begin{cases} \mathcal{L}_{11}\theta_1 + \mathcal{L}_{12}\theta_2 = f_1(\tau) - \sigma_1(\tau, \theta(\tau)),\ 0 \le \tau \le 1, \\ \mathcal{L}_{21}\theta_1 + \mathcal{L}_{22}\theta_2 = f_2(\tau) - \sigma_2(\tau, \theta(\tau)), \\ \theta_i(a) = \theta_i(b) = \gamma, \quad i = 1, 2, \end{cases} \tag{1.1}$$

where $\tau \in \Omega = [a,b] = [0,1]$, $\mathcal{L}_{i,j} : \mathcal{W}_2^3[0,1] \to \mathcal{W}_2^1[0,1],\ i,j = 1,2$ are linear operators, $\sigma_d(.,.)$ are nonlinear operators, $f_d(.)$ are given functions for $d = 1,\ 2$. Let $\theta(.) = (\theta_1(.), \theta_2(.))^T$ is unknown vector function which to be determined. In Equation (1.1), we suppose

$$\begin{cases} \mathcal{L}_{11}\theta_1 = \theta_1''(\tau) + a_1(\tau)\theta_1'(\tau) + a_2(\tau)\theta_1(\tau), \\ \mathcal{L}_{12}\theta_2 = \theta_2''(\tau) + a_3(\tau)\theta_2'(\tau) + a_4(\tau)\theta_2(\tau), \\ \mathcal{L}_{21}\theta_1 = \theta_1''(\tau) + b_1(\tau)\theta_1'(\tau) + b_2(\tau)\theta_1(\tau), \\ \mathcal{L}_{22}\theta_2 = \theta_2''(\tau) + b_3(\tau)\theta_2'(\tau) + b_4(\tau)\theta_2(\tau), \end{cases}$$

where $a_i(.),\ b_i(.)$ are given functions for $i = 1,2,3,4$. For matrix notation, we define the linear operator $\boldsymbol{L}$ as:

$$\boldsymbol{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}.$$

Also, we define $\boldsymbol{F} = (f_1, f_2)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$, so Equation (1.1) can be written in the following form

$$\begin{cases} \boldsymbol{L}(\theta(\tau)) = \boldsymbol{F}(\tau) - \boldsymbol{\sigma}(\tau, \theta(\tau)), \qquad 0 < \tau < 1, \\ \theta_i(a) = \theta_i(b) = \gamma, \quad i = 1, 2. \end{cases} \tag{1.2}$$

In the nonlinear case, we consider Equation (1.2) by the following iterative scheme

$$\boldsymbol{L}(\theta_n(\tau)) = \boldsymbol{F}(\tau) - \boldsymbol{\sigma}(\tau, \theta_{n-1}(\tau)), \qquad n = 2, 3, \ldots, \tag{1.3}$$

with $\boldsymbol{L}(\theta_1(\tau)) = \boldsymbol{F}(\tau)$, see [4] for more details.

2 Main idea

In this section, we defined the reproducing kernel space, their kernels and several theorems and lemmas.

In this paper, we consider the Hilbert space $\mathcal{W}_2^k[a,b] = \{\theta(.) | \theta^{(k-1)}(.)$ is absolutely continuous, $\theta^{(k)}(.) \in L^2[a,b], \theta(a) = \theta(b) = 0\}$,

$$\langle \theta(.), \alpha(.) \rangle_{\mathcal{W}_2^k} = \sum_{i=0}^{k-1} \theta^{(i)}(a)\ \alpha^{(i)}(a) + \int_a^b \theta^{(k)}(\tau)\alpha^{(k)}(\tau)d\tau,$$

$$\|\theta(.)\|_{\mathcal{W}_2^k} = \sqrt{\langle \theta, \theta \rangle_{\mathcal{W}_2^k}}, \quad \theta(.), \alpha(.) \in \mathcal{W}_2^k[a,b],$$

where k is a natural number. Also, we consider the Hilbert space

$$\boldsymbol{W}_2^{k_1,k_2}[a,b] = \mathcal{W}_2^{k_1}[a,b] \oplus \mathcal{W}_2^{k_2}[a,b],$$

with the inner product and norm

$$\langle \theta, \alpha \rangle_{\boldsymbol{W}_2^{k_1,k_2}} = \langle \theta_1, \alpha_1 \rangle_{\mathcal{W}_2^{k_1}} + \langle \theta_2, \alpha_2 \rangle_{\mathcal{W}_2^{k_2}}, \quad \|\theta\|_{\boldsymbol{W}_2^{k_1,k_2}} = \left(\sum_{i=1}^{2} \|\theta_i\|^2_{\mathcal{W}_2^{k_i}} \right)^{1/2},$$

where $\theta = (\theta_1, \theta_2)^T$, $\alpha = (\alpha_1, \alpha_2)^T$, $\theta_i, \alpha_i \in \mathcal{W}_2^{k_i}[a,b]$ for natural numbers k_i, and $i = 1, 2$. According to Equation (1.3), we have

$$\theta \in \boldsymbol{W}_2^{3,3}[0,1], \ \boldsymbol{F} - \boldsymbol{\sigma} \in \boldsymbol{W}_2^{1,1}[0,1].$$

Lemma 1. *If $\mathcal{L}_{i,j}$ in (1.1) are bounded linear operators, then $\boldsymbol{L} : \boldsymbol{W}_2^{3,3}[0,1] \to \boldsymbol{W}_2^{1,1}[0,1]$ is a bounded linear operator, where*

$$\boldsymbol{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix},$$

and the boundedness of $\mathcal{L}_{ij}$ implies that $\boldsymbol{L}$ is bounded, also the adjoint operator of $\boldsymbol{L}$ is

$$\boldsymbol{L}^* = \begin{pmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{12}^* \\ \mathcal{L}_{21}^* & \mathcal{L}_{22}^* \end{pmatrix},$$

where $\mathcal{L}_{ij}^*$ is the adjoint operator of $\mathcal{L}_{ij}$, [9].

Lemma 2. *The spaces $\mathcal{W}_2^3[0,1]$, $\mathcal{W}_2^2[0,1]$ and $\mathcal{W}_2^1[0,1]$ are reproducing kernel Hilbert space and their reproducing kernels are given as follows respectively,*

$$R_y(\tau) = \begin{cases} R(y,\tau), & \tau \ge y, \\ R(\tau,y), & \tau < y, \end{cases} \quad \bar{R}_y(\tau) = \begin{cases} \bar{R}(y,\tau), & \tau \ge y, \\ \bar{R}(\tau,y), & \tau < y, \end{cases}$$

$$\tilde{R}_y(\tau) = \begin{cases} \tilde{R}(y,\tau), & \tau \ge y, \\ \tilde{R}(\tau,y), & \tau < y, \end{cases}$$

where

$$\begin{aligned} R(y,\tau) &= y^5/120 + (\tau(864y - 720y^2 - 240y^3 - 36y^4 - 24y^5))/3744 \\ &\quad + (\tau^2(-360y + 378y^2 + 126y^3 + 15y^4 - 3y^5))/1872 + (\tau^3(-120y - 30y^2 \\ &\quad - 10y^3 + 5y^4 - y^5))/1872 + (\tau^5(-120y - 30y^2 - 10y^3 + 5y^4 - y^5))/18720 \\ &\quad + (\tau^4(120y + 30y^2 + 10y^3 - 5y^4 + y^5))/3744, \\ R(\tau,y) &= (\tau(864y - 720y^2 - 240y^3 + 120y^4 - 24y^5))/3744 + (\tau^2(-360y + 378y^2 \\ &\quad - 30y^3 + 15y^4 - 3y^5))/1872 + (\tau^5(156 - 120y - 30y^2 - 10y^3 + 5y^4 - y^5))/18720 \\ &\quad + (\tau^3(-120y + 126y^2 - 10y^3 + 5y^4 - y^5))/1872 \\ &\quad + (\tau^4(-36y + 30y^2 + 10y^3 - 5y^4 + y^5))/3744, \end{aligned}$$

$$\begin{aligned}&\bar{R}(y,\tau)=1/48\tau^3(-8+6y+3y^2-y^3)+1/16\tau^2(2y-3y^2+y^3)\\&\quad+1/16\tau(4y-6y^2+2y^3),\quad \bar{R}(\tau,y)=-(y^3/6)+1/48\tau^3(6y+3y^2-y^3)\\&\quad+1/16\tau^2(-6y-3y^2+y^3)+1/16\tau(4y+2y^2+2y^3),\\&\tilde{R}(y,\tau)=1+y,\quad \tilde{R}(\tau,y)=1+\tau.\end{aligned}$$

Let $\{\tau_l\}_{l=1}^{\infty}$ is a node set on $[0,1]$, we can deduce that:

$$\boldsymbol{\varphi}_{lj}(\tau)=\tilde{R}_\tau(\tau_l)\overrightarrow{e_j}=\begin{cases}(\tilde{R}_\tau(\tau_l),0)^T, & j=1,\\ \\ (0,\tilde{R}_\tau(\tau_l))^T, & j=2,\end{cases}$$

and $\boldsymbol{\psi}_{lj}(\tau)=\boldsymbol{L}^*\boldsymbol{\varphi}_{lj}(\tau)$ are reproducing kernels of $\boldsymbol{W}_2^{1,1}[0,1]$ and $\boldsymbol{W}_2^{3,3}[0,1]$, respectively, where $\overrightarrow{e_j}$ denotes the vector in $\mathbb{R}^2$ with 1 in the jth coordinate and 0's elsewhere, [4]. It can be proved that,

$$\left\langle\boldsymbol{\psi}_{si}(.),\boldsymbol{\psi}_{lj}(.)\right\rangle_{\boldsymbol{W}_2^{3,3}}=\begin{cases}0, & i\neq j,\\ \|R_{\tau_s}\|^2, & s=l,\ i=j,\\ R_{\tau_s}(\tau_l), & s\neq l,\ i=j.\end{cases}$$

Theorem 1. *For $j=1,2$ and $l=1,2,\ldots,$*

$$\boldsymbol{\psi}_{lj}(\tau)=\boldsymbol{L}R_{\tau_l}(\tau)\overrightarrow{e_j}.$$

Proof. By applying the reproducing properties [4],

$$\boldsymbol{\psi}_{lj}(\tau)=\left\langle\boldsymbol{L}^*\tilde{R}_\tau(.)\overrightarrow{e_j},R_{\tau_l}(.)\overrightarrow{e_j}\right\rangle_{\boldsymbol{W}_2^{3,3}}=\left\langle\tilde{R}_\tau(.)\overrightarrow{e_j},\boldsymbol{L}R_{\tau_l}(.)\overrightarrow{e_j}\right\rangle_{\boldsymbol{W}_2^{1,1}}=\boldsymbol{L}R_{\tau_l}(\tau)\overrightarrow{e_j}.$$

□

Theorem 2. *If $\{\tau_l\}_{l=1}^{\infty}$ is dense on $[0,1]$, then $\left\{\boldsymbol{\psi}_{lj}(\tau)\right\}_{(1,1)}^{(\infty,2)}$ is a complete function system in $\boldsymbol{W}_2^{3,3}[0,1]$.*

Proof. For each fixed $\theta(\tau)\in\boldsymbol{W}_2^{3,3}[0,1]$ if $\left\langle\theta(\tau),\boldsymbol{\psi}_{lj}(\tau)\right\rangle=0$, then

$$\begin{aligned}0&=\left\langle\theta(\tau),\boldsymbol{\psi}_{lj}(\tau)\right\rangle=\left\langle\theta(\tau),\boldsymbol{L}^*\boldsymbol{\varphi}_{lj}(\tau)\right\rangle=\left\langle\boldsymbol{L}\theta(\tau),\boldsymbol{\varphi}_{lj}(\tau)\right\rangle\\&=\left\langle\boldsymbol{L}\theta(\tau),\tilde{R}_\tau(\tau_l)\overrightarrow{e_j}\right\rangle=\boldsymbol{L}\theta(\tau_l).\end{aligned}$$

Taking into account the density of $\{\tau_l\}_{l=1}^{\infty}$, then $\theta(\tau_l)=0$. So, the proof is complete. □

Lemma 3. *For each fixed N, $\left\{\boldsymbol{\psi}_{lj}(\tau)\right\}_{(1,1)}^{(N,2)}$ is linearly independent in $\boldsymbol{W}_2^{3,3}[0,1]$, [11].*

Theorem 3. *If $\{\tau_s\}_{s=1}^{\infty}$ is dense on $[0,1]$ and the solution of Equation (1.2) is unique, then this solution is*

$$\theta(\tau)=\sum_{l=1}^{\infty}\sum_{j=1}^{2}c_{j,l}\psi_{lj}(\tau). \tag{2.1}$$

Proof. Substituting Equation (2.1) into Equation (1.2), then

$$\begin{aligned}\boldsymbol{L}\theta(\tau_s) &= \langle \boldsymbol{L}\theta(\tau), \boldsymbol{\varphi}_{si}(\tau)\rangle = \langle \theta(\tau), \boldsymbol{L}^*\boldsymbol{\varphi}_{si}(\tau)\rangle \\ &= \left\langle \sum_{l=1}^{\infty}\sum_{j=1}^{2} c_{j,l}\boldsymbol{\psi}_{lj}(\tau), \boldsymbol{\psi}_{si}(\tau)\right\rangle = \sum_{l=1}^{\infty}\sum_{j=1}^{2} c_{j,l}\left\langle \boldsymbol{\psi}_{lj}(\tau), \boldsymbol{\psi}_{si}(\tau)\right\rangle \\ &= \sum_{l=1}^{\infty}\sum_{j=1}^{2} c_{j,l}\boldsymbol{\psi}_{lj}(\tau_s) = \boldsymbol{F}(\tau_s) - \boldsymbol{\sigma}(\tau, \theta(\tau_s)).\end{aligned}$$

So, $\theta(\tau)$ is the solution of Equation (1.2), where $c_{j,l}$ for $j = 1, 2$ and $l = 1, \ldots$ are the unknown numbers to be determined. If Equation (1.2) is linear, then $\boldsymbol{\sigma} = 0$ and the approximate solution is

$$\theta_N(\tau) = \sum_{l=1}^{N}\sum_{j=1}^{2} c_{j,l}\psi_{lj}(\tau),$$

that $c_{j,l}$ for $j = 1, 2$ and $l = 1, \ldots, N$ are the unknown numbers to be determined and the proof is complete. □

In continuation we want to obtain the matrix notation for the unknowns in Equation (2.1). If Equation (1.2) is nonlinear, we give $\theta_1(.)$, n, N and the approximate solution is

$$\theta_{n,N}(\tau) = \sum_{l=1}^{N}\sum_{j=1}^{2} c_{j,l,n}\psi_{lj}(\tau), \quad n = 2, 3, \ldots, \tag{2.2}$$

where n is the number of the iteration for nonlinear term $\boldsymbol{\sigma}(\tau, \theta_{n-1,l}(\tau))$, N is number of collocation points on $[0, 1]$ and coefficients $c_{j,l,n}$ obtained as follows: substituting Equation (2.2) into Equation (1.2) and for sufficiently large N, we get $\boldsymbol{L}\theta_{n,N}(\tau) = \boldsymbol{F}(\tau) - \boldsymbol{\sigma}(\tau, \theta_{n-1,N}(\tau))$, according to the Theorem 3 , we can write

$$\sum_{l=1}^{N}\sum_{j=1}^{2} c_{j,l,n}\boldsymbol{\psi}_{lj}(\tau_s) = \boldsymbol{F}(\tau_s) - \boldsymbol{\sigma}(\tau, \theta_{n-1,N}(\tau_s)), \tag{2.3}$$

where $s = 1, 2, \ldots, N$ is number of collocation points. Now, using Theorem 1 we have

$$\begin{aligned}\sum_{l=1}^{N}\sum_{j=1}^{2} c_{j,l,n}\boldsymbol{\psi}_{lj}(\tau_s) &= \sum_{l=1}^{N} c_{1,l,n}\boldsymbol{\psi}_{l1}(\tau_s) + \sum_{l=1}^{N} c_{2,l,n}\boldsymbol{\psi}_{l2}(\tau_s) \\ &= \sum_{l=1}^{N} c_{1,l,n}(\boldsymbol{L}R_{\tau_l}(\tau_s)\overrightarrow{e_1}) + \sum_{l=1}^{N} c_{2,l,n}(\boldsymbol{L}R_{\tau_l}(\tau_s)\overrightarrow{e_2})\end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{L} \sum_{l=1}^{N} c_{1,l,n} R_{\tau_l}(\tau_s) \overrightarrow{e_1} + \boldsymbol{L} \sum_{l=1}^{N} c_{2,l,n} R_{\tau_l}(\tau_s) \overrightarrow{e_2} \\
&= \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} \begin{pmatrix} \sum\limits_{l=1}^{N} c_{1,l,n} R_{\tau_l}(\tau_s) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} \begin{pmatrix} 0 \\ \sum\limits_{l=1}^{N} c_{2,l,n} R_{\tau_l}(\tau_s) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L}_{11} \sum\limits_{l=1}^{N} c_{1,l,n} R_{\tau_l}(\tau_s) \\ \mathcal{L}_{21} \sum\limits_{l=1}^{N} c_{1,l,n} R_{\tau_l}(\tau_s) \end{pmatrix} + \begin{pmatrix} \mathcal{L}_{12} \sum\limits_{l=1}^{N} c_{2,l,n} R_{\tau_l}(\tau_s) \\ \mathcal{L}_{22} \sum\limits_{l=1}^{N} c_{2,l,n} R_{\tau_l}(\tau_s) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L}_{11} \sum\limits_{l=1}^{N} c_{1,l,n} R_{\tau_l}(\tau_s) + \mathcal{L}_{12} \sum\limits_{l=1}^{N} c_{2,l,n} R_{\tau_l}(\tau_s) \\ \mathcal{L}_{21} \sum\limits_{l=1}^{N} c_{1,l,n} R_{\tau_l}(\tau_s) + \mathcal{L}_{22} \sum\limits_{l=1}^{N} c_{2,l,n} R_{\tau_l}(\tau_s) \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{l=1}^{N} \mathcal{L}_{11} R_{\tau_l}(\tau_s) & \sum\limits_{l=1}^{N} \mathcal{L}_{12} R_{\tau_l}(\tau_s) \\ \sum\limits_{l=1}^{N} \mathcal{L}_{21} R_{\tau_l}(\tau_s) & \sum\limits_{l=1}^{N} \mathcal{L}_{22} R_{\tau_l}(\tau_s) \end{pmatrix} \begin{pmatrix} c_{1,l,n} \\ c_{2,l,n} \end{pmatrix}.
\end{aligned}
$$

Also,

$$
\boldsymbol{F}(\tau_s) - \boldsymbol{\sigma}(\tau, \theta_{n-1,N}(\tau_s)) = \begin{pmatrix} f_1(\tau_s) \\ f_2(\tau_s) \end{pmatrix} - \begin{pmatrix} \sigma_1(\tau, \theta_{n-1,N}(\tau_s)) \\ \sigma_2(\tau, \theta_{n-1,N}(\tau_s)) \end{pmatrix}.
$$

So, according to Equation (2.3) we can deduce that

$$
\begin{aligned}
\begin{pmatrix} \sum\limits_{l=1}^{N} \mathcal{L}_{11} R_{\tau_l}(\tau_s) & \sum\limits_{l=1}^{N} \mathcal{L}_{12} R_{\tau_l}(\tau_s) \\ \sum\limits_{l=1}^{N} \mathcal{L}_{21} R_{\tau_l}(\tau_s) & \sum\limits_{l=1}^{N} \mathcal{L}_{22} R_{\tau_l}(\tau_s) \end{pmatrix} \begin{pmatrix} c_{1,l,n} \\ c_{2,l,n} \end{pmatrix} = \begin{pmatrix} f_1(\tau_s) \\ f_2(\tau_s) \end{pmatrix} \\
- \begin{pmatrix} \sigma_1(\tau, \theta_{n-1,N}(\tau_s)) \\ \sigma_2(\tau, \theta_{n-1,N}(\tau_s)) \end{pmatrix}.
\end{aligned} \tag{2.4}
$$

Then,

$$
\begin{aligned}
&\boldsymbol{A} = \begin{pmatrix} \sum\limits_{l=1}^{N} \mathcal{L}_{11} R_{\tau_l}(\tau_s) & \sum\limits_{l=1}^{N} \mathcal{L}_{12} R_{\tau_l}(\tau_s) \\ \sum\limits_{l=1}^{N} \mathcal{L}_{21} R_{\tau_l}(\tau_s) & \sum\limits_{l=1}^{N} \mathcal{L}_{22} R_{\tau_l}(\tau_s) \end{pmatrix}, \\
&\boldsymbol{C} = \begin{pmatrix} c_{1,l,n} \\ c_{2,l,n} \end{pmatrix}, \boldsymbol{F} = \begin{pmatrix} f_1(\tau_s) \\ f_2(\tau_s) \end{pmatrix} - \begin{pmatrix} \sigma_1(\tau, \theta_{n-1,N}(\tau_s)) \\ \sigma_2(\tau, \theta_{n-1,N}(\tau_s)) \end{pmatrix},
\end{aligned}
$$

therefore, we can write $\boldsymbol{A}\ \boldsymbol{C} = \boldsymbol{F}$, finally, according to the Lemma 3 $\boldsymbol{A}^{-1}$ is exist and $\boldsymbol{C} = \boldsymbol{A}^{-1}\ \boldsymbol{F}$.

3 Error estimation

Lemma 4. *Let* $S = \left\{\theta(.) = (\theta_1(.), \theta_2(.)) \mid \|\theta\|_{\boldsymbol{W}_2^{3,3}} \leq \lambda\right\}$ *is a compact set in space* $C^2[0,1]$, *where* λ *is a constant, [19].*

Lemma 5. *Assume in system* (1.2), $\|\theta\|_{\boldsymbol{W}_2^{3,3}}$ *is bounded,* $\{\tau_s\}_{s=1}^{\infty}$ *is dense set on* $[0,1]$, $\boldsymbol{L}(\theta(.))$ *is an invertible continuous function of* $\theta(.)$, *and* $\boldsymbol{\sigma}(.,\theta(.))$ *is a continuous function of* $\theta(.)$, *then the exact solution* $\theta(.)$ *and the approximate solution* $\theta_{n,N}(.)$ *for Equation* (1.2) *are existent, [19].*

Theorem 4. *If* $\theta(.) = (\theta_1(.), \theta_2(.)) \in \boldsymbol{W}_2^{3,3}$ *is the solution of Equation* (1.2) *then the approximate solution* $\theta_{n,N}(.) = (\theta_{1,n,N}(.), \theta_{2,n,N}(.))$ *and its derivative*

$$\theta^{(i)}_{n,N}(.) = (\theta^{(i)}_{1,n,N}(.), \theta^{(i)}_{2,n,N}(.)),$$

converges uniformly to $\theta(.)$ *and* $\theta^{(i)}(.)$, *respectively, for* $i = 1, 2$.

Proof. By the summation of the two equations in Equation (1.1), we have

$$\theta_1''(\tau) + c(\tau)\theta_1'(\tau) + d(\tau)\theta_1(\tau) + \sigma_1(\tau, \theta(\tau)) = f(\tau), \tag{3.1}$$

where $c(.), d(.), f(.)$ and $\sigma_1(.)$ are known functions. Clearly, Equation (3.1) is a nonlinear equation in reproducing kernel space $\mathcal{W}_2^k[0,1]$, In addition, from Lemma 5 we know that $\theta_{1,n,N}(.)$ is the approximate solution of $\theta_1(.)$, then

$$\begin{aligned} |\theta_1(\tau) - \theta_{1,n,N}(\tau)| &= |\langle \theta_1 - \theta_{1,n,N}, R_\tau\rangle| \leq \|\theta_1 - \theta_{1,n,N}\|_{\mathcal{W}_2^k} \|R_\tau\|_{\mathcal{W}_2^k} \\ &\leq P_1 \|\theta_1 - \theta_{1,n,N}\|_{\mathcal{W}_2^k}. \end{aligned}$$

Also, for first derivative we have

$$\begin{aligned} |\theta_1^{'}(\tau) - \theta_{1,n,N}^{'}(\tau)| &= |\frac{\partial}{\partial\tau}(\langle \theta_1 - \theta_{1,n,N}, R_\tau\rangle_{\mathcal{W}_2^k})| = \left| \left\langle \theta_1 - \theta_{1,n,N}, \frac{\partial}{\partial \tau} R_\tau \right\rangle_{\mathcal{W}_2^k} \right| \\ &\leq \|\theta_1 - \theta_{1,n,N}\|_{\mathcal{W}_2^k} \left\| \frac{\partial}{\partial\tau} R_\tau \right\|_{\mathcal{W}_2^k} \leq Z_1 \|\theta_1 - \theta_{1,n,N}\|_{\mathcal{W}_2^k}, \end{aligned}$$

where P_1 and Z_1 are constants. Similarly, we have

$$\begin{aligned} |\theta_2(\tau) - \theta_{2,n,N}(\tau)| &\leq P_2 \|\theta_2 - \theta_{2,n,N}\|_{\mathcal{W}_2^k}, \\ |\theta_2^{'}(\tau) - \theta_{2,n,N}^{'}(\tau)| &\leq Z_2 \|\theta_2 - \theta_{2,n,N}\|_{\mathcal{W}_2^k}. \end{aligned}$$

□

Theorem 5. *Let* $\theta(.) = (\theta_1(.), \theta_2(.))$ *and* $\theta_{n,N}(.) = (\theta_{1,n,N}(.), \theta_{2,n,N}(.))$ *be the exact and approximate solution of Equation* (1.2), *respectively. If* $\theta \in C^3[0,1]$, $\theta_{n,N} \in \boldsymbol{W}_2^{3,3}[0,1]$ *and* $\left\|\theta''_{i,n,N}\right\|_\infty \leq M_i$, $i = 1, 2$, *then for* $j = 1, 2$

$$\|\theta_j - \theta_{j,n,N}\|_\infty \leq C_j h^3, \quad \left\|\theta_j' - \theta_{j,n,N}'\right\|_\infty \leq C_j h^2,$$

where C_j *is a constant, [1].*

Remark 1. According to the Lemma 3, $\boldsymbol{A}^{-1}$ exists then the solution of Equation (1.2) is exist and unique. Furthermore, we can conclude that the present method is stable in $\boldsymbol{W}_2^{3,3}[0,1]$.

Remark 2. In the present method, the choice of space depends on the type of problem. For example, in Examples 2 and 3, the highest derivative order for θ_1 and θ_2 is equal to 2. As a result, $\theta_1, \theta_2 \in \mathcal{W}_2^3[0,1]$, and the appropriate space for these two examples is $\boldsymbol{L} : \boldsymbol{W}_2^{3,3}[0,1] \to \boldsymbol{W}_2^{1,1}[0,1]$. But in Example 1, the highest order of the derivative for θ_1 and θ_2 is 2 and 1, respectively. As a consequence, $\theta_1 \in \mathcal{W}_2^3[0,1]$ and $\theta_2 \in \mathcal{W}_2^2[0,1]$, so the appropriate space for Example 1 is $\boldsymbol{L} : \boldsymbol{W}_2^{3,2}[0,1] \to \boldsymbol{W}_2^{1,1}[0,1]$. Of course, it is clear that with the change of space, the convergence order will also change. Therefore, in Example 1, the convergence order of θ_2 and θ_2' will be as follows

$$\|\theta_2 - \theta_{2,n,N}\|_\infty \le C_2 h^2, \quad \left\|\theta_2' - \theta_{2,n,N}'\right\|_\infty \le C_2 h.$$

Remark 3. In this paper, the convergence formulas can be obtained:

$$C.F_i = \log_2 \frac{\|\theta_i - \theta_{i,n,N}\|_\infty}{\|\theta_i - \theta_{i,n,2N}\|_\infty}, \quad C.F_i' = \log_2 \frac{\|\theta_i' - \theta_{i,n,N}'\|_\infty}{\|\theta_i' - \theta_{i,n,2N}'\|_\infty},$$

where $i = 1, 2$.

4 Numerical results

In this section, we solve three numerical examples by the present method. We show that the present method (PM) is better than the methods of [8] and [9].

Example 1. [8] Consider the non-linear system of the BVP

$$\begin{cases} \theta_1''(\tau) + \tau\theta_1(\tau) + 2\tau\theta_2(\tau) + \tau\theta_1^2(\tau) = f(\tau), \qquad 0 < \tau < 1, \\ \theta_2'(\tau) + \theta_2(\tau) + \tau^2\theta_1(\tau) + \sin(\tau)\theta_2^2(\tau) = g(\tau), \\ \theta_1(0) = \theta_1(1) = 0, \qquad \theta_2(0) = \theta_2(1) = 0, \end{cases}$$

where $\theta(\tau) = [\tau - \tau^2, \sin(\pi\tau)]$. We solved this example using Equation (2.4) in $\boldsymbol{W}_2^{3,2}[0,1]$ and compared the present method to other methods in Tables 1 and 2. The numerical results for the convergence order are given in Table 3.

Example 2. Consider the non-linear system of the BVP

$$\begin{cases} \theta_1''(\tau) + \tau\theta_1'(\tau) + \cos(\pi\tau)\theta_2'(\tau) = f(\tau), \qquad 0 < \tau < 1, \\ \theta_2''(\tau) + \tau\theta_1'(\tau) + \tau\theta_1^2(\tau) = g(\tau), \\ \theta_1(0) = \theta_1(1) = 0, \qquad \theta_2(0) = \theta_2(1) = 0, \end{cases}$$

where $\theta(\tau) = [(\tau - 1)\sin(\pi\tau), \tau - \tau^2]$. We solved this example using Equation (2.4) in $\boldsymbol{W}_2^{3,3}[0,1]$ and Table 4 shows that the absolute error of the present method and other methods, while Table 5 shows the convergence orders. In Figures 1 and 2 we compared the absolute errors in the present method and method in [8] via the independent variable τ.

Table 1. Example 1: Absolute errors for θ_1.

τ	[9] θ_1	[5] θ_1	[8] $\theta_{1,5,21}$	[8] $\theta_{1,5,51}$	PM $\theta_{1,5,21}$	PM $\theta_{1,5,51}$
0.08	5.0×10^{-4}	1.4×10^{-4}	7.7×10^{-5}	2.0×10^{-5}	1.2×10^{-4}	1.6×10^{-6}
0.24	1.4×10^{-3}	4.4×10^{-5}	2.2×10^{-4}	5.7×10^{-5}	1.1×10^{-4}	5.2×10^{-7}
0.40	2.1×10^{-3}	6.7×10^{-5}	3.3×10^{-4}	8.6×10^{-5}	9.7×10^{-5}	5.7×10^{-7}
0.56	2.2×10^{-3}	9.3×10^{-5}	3.7×10^{-4}	9.8×10^{-5}	7.5×10^{-5}	1.0×10^{-6}
0.72	1.8×10^{-3}	4.9×10^{-5}	3.1×10^{-4}	9.4×10^{-5}	4.8×10^{-5}	8.6×10^{-7}
0.88	9.0×10^{-4}	8.6×10^{-5}	1.5×10^{-4}	6.5×10^{-5}	2.1×10^{-5}	3.9×10^{-7}
0.96	3.0×10^{-4}	7.1×10^{-5}	5.4×10^{-5}	1.4×10^{-5}	6.9×10^{-6}	1.3×10^{-7}

Table 2. Example 1: Absolute errors for θ_2.

τ	[9] θ_2	[5] θ_2	[8] $\theta_{2,5,21}$	[8] $\theta_{2,5,51}$	PM $\theta_{2,5,21}$	PM $\theta_{1,5,51}$
0.08	2.0×10^{-3}	2.4×10^{-4}	7.1×10^{-4}	1.1×10^{-4}	4.2×10^{-4}	6.5×10^{-6}
0.24	5.6×10^{-3}	2.3×10^{-3}	1.9×10^{-3}	3.3×10^{-4}	7.2×10^{-4}	3.3×10^{-5}
0.40	7.9×10^{-3}	8.9×10^{-4}	2.7×10^{-3}	4.6×10^{-4}	2.4×10^{-4}	5.1×10^{-6}
0.56	8.2×10^{-3}	1.4×10^{-3}	2.8×10^{-3}	4.8×10^{-4}	3.4×10^{-4}	1.9×10^{-5}
0.72	6.5×10^{-3}	3.1×10^{-3}	2.2×10^{-3}	3.8×10^{-4}	2.0×10^{-4}	4.0×10^{-5}
0.88	3.1×10^{-3}	1.6×10^{-3}	1.7×10^{-3}	2.9×10^{-4}	8.6×10^{-5}	5.0×10^{-7}
0.96	1.0×10^{-3}	9.8×10^{-4}	3.6×10^{-4}	6.2×10^{-5}	8.4×10^{-5}	2.6×10^{-5}

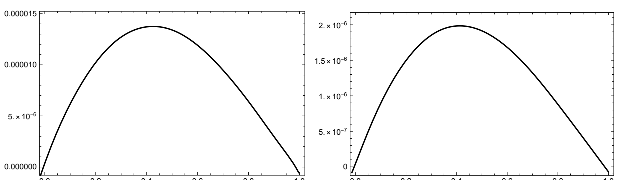

Figure 1. Figures of absolute errors for Example 2 with Method [8] (Left:$|\theta_2(\tau) - \theta_{2,5,21}(\tau)|$; Right: $|\theta_2(\tau) - \theta_{2,5,51}(\tau)|$).

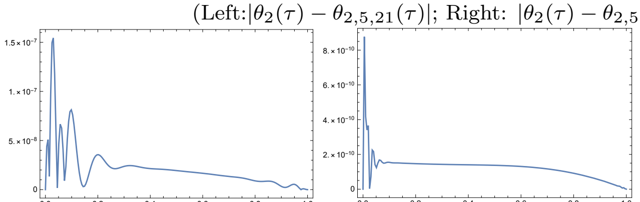

Figure 2. Figures of absolute errors for Example 2 with Present Method (Left:$|\theta_2(\tau) - \theta_{2,5,21}(\tau)|$; Right: $|\theta_2(\tau) - \theta_{2,5,51}(\tau)|$).

Table 3. Max absolute error and convergence order.

	$N = 5$	$N = 10$	$N = 20$	$N = 40$
Example 1				
$\Vert\theta_1 - \theta_{1,5,N}\Vert_\infty$	7.0×10^{-3}	1.70×10^{-3}	1.50×10^{-4}	9.0×10^{-6}
$C.F_1$	-	2.04182	3.5025	4.07889
$\Vert\theta_2 - \theta_{2,5,N}\Vert_\infty$	2.20×10^{-2}	6.40×10^{-3}	1.10×10^{-3}	2.60×10^{-4}
$C.F_2$	-	1.78136	2.54057	2.08092
$\Vert\theta_1' - \theta_{1,5,N}'\Vert_\infty$	7.30×10^{-2}	2.80×10^{-2}	5.60×10^{-3}	7.50×10^{-4}
$C.F_1'$	-	1.38247	2.32193	2.90046
$\Vert\theta_2' - \theta_{2,5,N}'\Vert_\infty$	3.40×10^{-1}	1.80×10^{-1}	7.50×10^{-2}	3.60×10^{-2}
$C.F_2'$	-	0.917538	1.26303	1.05889

Table 4. Example 2: Absolute errors for θ_1.

τ	[14]	[8]	[8]	PM	PM
	θ_1	$\theta_{1,5,21}$	$\theta_{1,5,51}$	$\theta_{1,5,21}$	$\theta_{1,5,51}$
0.00	0.00	0.00	0.00	0.00	0.00
0.10	3.0×10^{-4}	4.4×10^{-5}	7.1×10^{-6}	3.9×10^{-8}	2.2×10^{-10}
0.30	7.8×10^{-3}	1.0×10^{-4}	1.6×10^{-5}	1.2×10^{-8}	4.5×10^{-10}
0.50	2.7×10^{-2}	1.2×10^{-4}	1.8×10^{-5}	3.4×10^{-9}	4.8×10^{-10}
0.70	4.6×10^{-2}	9.6×10^{-5}	1.5×10^{-5}	1.2×10^{-8}	2.7×10^{-10}
0.90	3.1×10^{-2}	3.8×10^{-5}	6.0×10^{-6}	9.4×10^{-8}	5.2×10^{-11}
1.00	0.00	0.00	0.00	0.00	0.00

Table 5. Max absolute error and convergence order.

	$N = 5$	$N = 10$	$N = 20$	$N = 40$
Example 2				
$\Vert\theta_1 - \theta_{1,5,N}\Vert_\infty$	3.10×10^{-4}	1.20×10^{-5}	2.60×10^{-7}	3.40×10^{-9}
$C.F_1$	-	4.69116	5.52838	6.25683
$\Vert\theta_2 - \theta_{2,5,N}\Vert_\infty$	2.30×10^{-4}	9.50×10^{-6}	2.10×10^{-7}	2.70×10^{-9}
$C.F_2$	-	4.59756	5.49947	6.28129
$\Vert\theta_1' - \theta_{1,5,N}'\Vert_\infty$	3.50×10^{-3}	3.40×10^{-4}	2.70×10^{-5}	1.70×10^{-6}
$C.F_1'$	-	3.36375	3.6545	3.98935
$\Vert\theta_2' - \theta_{2,5,N}'\Vert_\infty$	2.70×10^{-3}	2.80×10^{-4}	2.10×10^{-5}	1.40×10^{-6}
$C.F_2'$	-	3.26946	3.73697	3.90689

Example 3. [8] Consider the non-linear system of the BVP

$$\begin{cases} \theta_1''(\tau) + 20\theta_1'(\tau) + 4\cos(\tau)\theta_1(\tau) + \sin(\theta_1(\tau)\theta_2(\tau)) = f(\tau), \quad 0 < \tau < 1, \\ \theta_2''(\tau) + 5e^\tau\theta_2'(\tau) + 6\sinh(\tau)\theta_2(\tau) + \cos(\theta_2(\tau)) = g(\tau), \\ \theta_1(0) = 1, \, \theta_1(1) = e, \qquad \theta_2(0) = 0, \, \theta_2(1) = \sinh(1), \end{cases}$$

where $\theta(\tau) = [e^\tau, \sinh(\tau)]$. Let $\tilde{\theta_1} = \theta_1 + c_0 + c_1 x$, $\tilde{\theta_2} = \theta_2 + d_0 + d_1 x$, where c_0, c_1, d_0, d_1 are determined by letting $\tilde{\theta_1}(0) = \tilde{\theta_1}(1) = 0$,$\tilde{\theta_2}(0) = \tilde{\theta_2}(1) = 0$. We solved this example using Equation (2.4) in $\boldsymbol{W}_2^{3,3}[0,1]$. In Figures 3–6

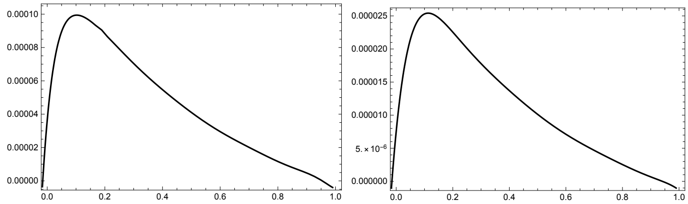

Figure 3. Figures of relative errors for Example 3 with Method [8] (Left:$|\frac{\theta_1(\tau)-\theta_{1,5,21}(\tau)}{\theta_1(\tau)}|$; Right: $|\frac{\theta_1(\tau)-\theta_{1,5,51}(\tau)}{\theta_1(\tau)}|$).

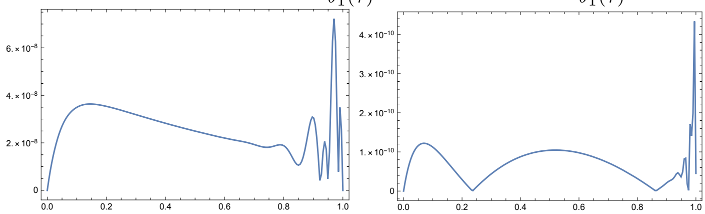

Figure 4. Figures of relative errors for Example 3 with Present Method (Left:$|\frac{\theta_1(\tau)-\theta_{1,5,21}(\tau)}{\theta_1(\tau)}|$; Right: $|\frac{\theta_1(\tau)-\theta_{1,5,51}(\tau)}{\theta_1(\tau)}|$).

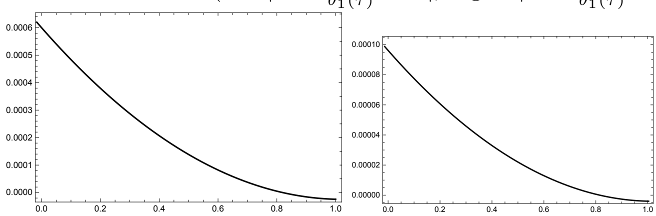

Figure 5. Figures of relative errors for Example 3 with Method [8] (Left:$|\frac{\theta_2(\tau)-\theta_{2,5,21}(\tau)}{\theta_2(\tau)}|$; Right: $|\frac{\theta_2(\tau)-\theta_{2,5,51}(\tau)}{\theta_2(\tau)}|$).

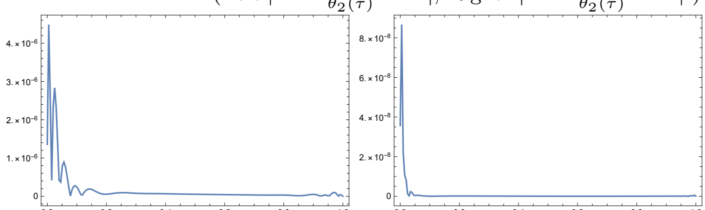

Figure 6. Figures of relative errors for Example 3 with Present Method (Left:$|\frac{\theta_2(\tau)-\theta_{2,5,21}(\tau)}{\theta_2(\tau)}|$; Right: $|\frac{\theta_2(\tau)-\theta_{2,5,51}(\tau)}{\theta_2(\tau)}|$).

we compared the relative errors in the present method and method in [8] via the independent variable τ. The convergence order and maximum errors for Figures 1–6 are shown in Tables 6 and 7.

Table 6. Max absolute error and convergence order.

	$N=5$	$N=10$	$N=20$	$N=40$
Example 3				
$\|\theta_1-\theta_{1,5,N}\|_\infty$	3.60×10^{-4}	1.0×10^{-5}	2.40×10^{-7}	3.80×10^{-9}
$C.F_1$	-	5.16993	5.38082	5.98089
$\|\theta_2-\theta_{2,5,N}\|_\infty$	2.40×10^{-4}	5.20×10^{-6}	1.50×10^{-7}	2.10×10^{-9}
$C.F_2$	-	5.52838	5.11548	6.15843
$\|\theta'_1-\theta'_{1,5,N}\|_\infty$	4.60×10^{-3}	3.50×10^{-4}	2.60×10^{-5}	1.90×10^{-6}
$C.F'_1$	-	3.71621	3.75077	3.77444
$\|\theta'_2-\theta'_{2,5,N}\|_\infty$	2.10×10^{-3}	1.90×10^{-4}	1.50×10^{-5}	1.10×10^{-6}
$C.F'_2$	-	3.46632	3.66297	3.76939

Table 7. Max absolute errors for Figures 1–2 and Max relative errors for Figures 3–6.

	Method in [8]	Present Method
Example 2		
$\|\theta_1-\theta_{1,5,21}\|_\infty$	1.40×10^{-5}	1.66×10^{-7}
$\|\theta_2-\theta_{2,5,51}\|_\infty$	2.0×10^{-6}	8.70×10^{-10}
Example 3		
$\max_\tau \left\|\frac{\theta_1(\tau)-\theta_{1,5,21}(\tau)}{\theta_1(\tau)}\right\|$	1.0×10^{-4}	7.20×10^{-8}
$\max_\tau \left\|\frac{\theta_1(\tau)-\theta_{1,5,51}(\tau)}{\theta_1(\tau)}\right\|$	2.50×10^{-5}	4.40×10^{-10}
$\max_\tau \left\|\frac{\theta_2(\tau)-\theta_{2,5,21}(\tau)}{\theta_2(\tau)}\right\|$	6.20×10^{-4}	4.50×10^{-6}
$\max_\tau \left\|\frac{\theta_2(\tau)-\theta_{2,5,51}(\tau)}{\theta_2(\tau)}\right\|$	1.0×10^{-4}	8.60×10^{-8}

5 Conclusions

In this paper, we introduced a new method based on RKM without the Gram-Schmidt orthogonalization process to solve systems of second-order BVPs. The numerical results verified that the present method is better than of the method in [5,8,9,14]. In addition, we solved Example 1 in $\boldsymbol{W}_2^{3,2}[0,1]$ space and Examples 2 and 3 in $\boldsymbol{W}_2^{3,3}[0,1]$ space. According to Tables 4–6 and the maximum absolute error, it can be concluded that when bases are selected from two different spaces, as in Example 1, the method is less accurate compared to when they are selected from one space, as in Examples 2 and 3. Finally, according to the flexibility in the selection of points, spaces, and bases in the present method, the suitable strategy can be adopted to solve the different problems.

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References

[1] E. Babolian, S. Javadi and E. Moradi. Error analysis of reproducing kernel Hilbert space method for solving functional integral equations. *Comput. Appl. Math.*, **300**:300–311, 2016. https://doi.org/10.1016/j.cam.2016.01.008.

[2] M. Baccouch, H. Temimi and M. Ben-Romdhane. A discontinuous Galerkin method for systems of stochastic differential equations with applications to population biology, finance, and physics. *Comput. Appl. Math.*, **388**:113297, 2021. https://doi.org/10.1016/j.cam.2020.113297.

[3] M. Cakmak and S. Alkan. A numerical method for solving a class of systems of nonlinear Pantograph differential equations. *Alexandria Engineering*, **61**:2651–2661, 2022. https://doi.org/10.1016/j.aej.2021.07.028.

[4] M.G. Cui and Y. Lin. *Nonlinear Numerical Analysis in the Reproducing Kernel Space, Nova Science, Hauppauge.* Inc., Hauppauge, 2009.

[5] M. Dehghan and A. Saadatmandi. The numerical solution of a nonlinear system of second-order boundary value problems using the sinc-collocation method. *Math. Comput. Model.*, **46**:1434–1441, 2007. https://doi.org/10.1016/j.mcm.2007.02.002.

[6] A. Faghih and P. Mokhtary. A novel Petrov-Galerkin method for a class of linear systems of fractional differential equations. *Appl. Num. Math.*, **169**:396–414, 2021. https://doi.org/10.1016/j.apnum.2021.07.012.

[7] M. Faheem, A. Khan and P.J.Y. Wong. A Legendre wavelet collocation method for 1D and 2D coupled time-fractional nonlinear diffusion system. *Comput. Math. Appl.*, **128**:214–238, 2022. https://doi.org/10.1016/j.camwa.2022.10.014.

[8] F. Geng and M. Cui. Homotopy perturbation–reproducing kernel method for nonlinear systems of second order boundary value problems. *Comput. Appl. Math.*, **235**:2405–2411, 2011. https://doi.org/10.1016/j.cam.2010.10.040.

[9] F.Z. Geng and M.G. Cui. Solving a nonlinear system of second order boundary value problems. *J. Math. Anal. Appl.*, **327**:1167–1181, 2007. https://doi.org/10.1016/j.jmaa.2006.05.011.

[10] F.A. Ghassabzadeh, E. Tohidi and H. Singh. RBF collocation approach to calculate numerically the solution of the nonlinear system of qFDEs. *King Saud University-Science*, **33**:101288, 2021. https://doi.org/10.1016/j.jksus.2020.101288.

[11] V.L. Hansen. *Functional Analysis, Entering Hilbert Space.* World Scientific Publishing Co. Pte. Ltd., 2006. https://doi.org/10.1142/5976.

[12] W. Jiang and Z. Chen. Solving a system of linear Volterra integral equations using the new reproducing kernel method. *Applied Mathematics and Computation*, **219**:10225–10230, 2013. https://doi.org/10.1016/j.amc.2013.03.123.

[13] B. Liu, Y. Wu, J. Guo, H. Zhang, J. Niu and F. Li. Finite difference Jacobian based Newton-Krylov coupling method for solving multi-physics nonlinear system of nuclear reactor. *Annals of Nuclear Energy*, **148**:107670, 2020. https://doi.org/10.1016/j.anucene.2020.107670.

[14] J.F. Lu. Variational iteration method for solving a nonlinear system of second-order boundary value problems. *Comput. Math. Appl.*, **54**:1133–1138, 2007. https://doi.org/10.1016/j.camwa.2006.12.060.

[15] J. Niu, M.Q. Xu, Y.Z. Lin and Q. Xue. Numerical solution of nonlinear singular boundary value problems. *Comput. Appl. Math.*, **333**:42–51, 2018. https://doi.org/10.1016/j.cam.2017.09.040.

[16] J. Niu and J. Zhang. Lobatto-reproducing kernel method for solving a linear system of second order boundary value problems. *Appl. Math. Comput*, **68**:3631–3653, 2022. https://doi.org/10.1007/s12190-021-01685-9.

[17] R. Saadeh. Numerical algorithm to solve a coupled system of fractional order using a novel reproducing kernel method. *Alexandria Engineering*, **60**:4583–4591, 2021. https://doi.org/10.1016/j.aej.2021.03.033.

[18] H. Sahihi, T. Allahviranloo and S. Abbasbandy. Solving system of second-order bvps using a new algorithm based on reproducing kernel Hilbert space. *Appl. Num. Math.*, **151**:27–39, 2020. https://doi.org/10.1016/j.apnum.2019.12.008.

[19] Y. Wang, T. Chaolu and Z. Chen. Using reproducing kernel for solving a class of singular weakly nonlinear boundary value problems. *Comput. Appl. Math.*, **87**:367–380, 2010. https://doi.org/10.1080/00207160802047640.

[20] Y. Yu, J. Niu, J. Zhang and S.Y. Ning. A reproducing kernel method for nonlinear C-q-fractional IVPs. *Appl. Math. Lett*, **125**:107751J, 2022. https://doi.org/10.1016/j.aml.2021.107751.