Mathematical Modelling and Analysis
Volume 25, Issue 2, 226–240, 2020
https://doi.org/10.3846/mma.2020.10383
http://mma.vgtu.lt
ISSN: 1392-6292
eISSN: 1648-3510

Decay Rates for a Coupled Viscoelastic Lamé System with Strong Damping

Baowei Feng[a] **and Haiyan Li**[b]

[a] *Department of Economic Mathematics, Southwestern University of Finance and Economics*
Chengdu 611130, China
[b] *College of Mathematics and Information Science, North Minzu University*
Yinchuan 750021, China
E-mail(*corresp.*): bwfeng@swufe.edu.cn
E-mail: lihaiyanmath@163.com

Received May 24, 2019; revised January 19, 2020; accepted January 19, 2020

Abstract. In [6] Beniani, Taouaf and Benaissa studied a coupled viscoelastic Lamé system with strong dampings and established a general decay result. In this paper, we continue to study the system. Assuming $g_i'(t) \leq -\xi_i(t)H_i(g_i(t))$, $i = 1, 2$, we establish an explicit and general decay result, which is optimal, to the system. This result improves earlier results in [6].

Keywords: Lamé system, energy decay, viscoelastic damping, convexity.

AMS Subject Classification: 35B40; 93D20.

1 Introduction

In [6], Beniani, Taouaf and Benaissa considered the following coupled viscoelastic Lamé system with strong dampings

$$u_{tt} + \alpha v - \Delta_e u + \int_0^t g_1(t-s)\Delta u(s)ds - \mu_1 \Delta u_t = 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$v_{tt} + \alpha u - \Delta_e v + \int_0^t g_2(t-s)\Delta v(s)ds - \mu_2 \Delta v_t = 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.2}$$

$$u(x,t) = v(x,t) = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{1.3}$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ u_t(x,0) = u_1(x),\ v_t(x,0) = v_1(x), \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. The constant α is coupling constant, and μ_1, μ_2 are two positive constants. The relaxation functions $g_1(t)$ and $g_2(t)$ are real functions. The elasticity operator Δ_e is the 3×1 matrix-valued differential operator, which is given by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu)\nabla(\mathrm{div} u), \quad u = (u_1, u_2, u_3)^{tr},$$

and μ and λ are the Lamé constants satisfying the following conditions

$$\mu > 0, \quad \lambda + \mu \geq 0.$$

In [6], the authors proved the well-posedness of solutions to problem (1.1)–(1.4). Under the assumptions on $g_i(t)$

$$g_i'(t) \leq -\xi_i(t) g_i(t), \quad i = 1, 2, \ \forall \ t > 0,$$

they established the general decay rates of energy of the form

$$E(t) \leq c e^{-\gamma \int_0^t \xi(s) ds}.$$

In this paper, we continue to consider problem (1.1)–(1.4), and improve the energy decay results in [6] to establish explicit and general energy decay results for a wider class of relaxation function.

For a single Lamé equation, Bchatnia and Daoulatli [4] considered a Lamé system with localized nonlinear damping

$$u_{tt} - \Delta_e u + a(x) g(u_t) = f(x),$$

and established a general decay result of energy. Beniani et al. [7] studied energy decay of a time-delayed Lamé system. With respect to Lamé system with viscoelastic term, Bchatnia and Guesmia [5] investigated the system with past history

$$u_{tt} - \Delta_e u + \int_0^\infty g(s) \Delta u(t-s) ds = 0,$$

and obtained a more general energy decay. When $\lambda + \mu = 0$, the Lamé system reduces to classical wave system. For the following wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s) \Delta u(s) ds = \mathcal{F}(u), \tag{1.5}$$

Messaoudi [17, 18], by taking $\mathcal{F} = 0$ and $\mathcal{F} = |u|^\gamma u, \ \gamma > 0$, respectively and assuming $g'(t) \leq -\xi(t) g(t)$, obtained general decay results. We also mention Han and Wang [10], Liu [14, 15], Messaoudi and Mustafa [16], Mustafa [24] and Park and Park [28], where the authors get general decay of energy for problems related to (1.5) use this assumption on g. Lasiecka et al. [11] considered another general assumption on g: $g'(t) \leq -H(g(t))$, where H is strictly convex and increasing function and was first introduced by Alabau-Boussouira and Cannarsa [2]. After that, there are some stability results established by using this condition. See Cavalcanti et al. [8, 9], Lasiecka et al. [13], Mustafa [23], Mustafa and Messaoudi [27] and Xiao and Liang [30]. Very recently, in [25, 26], Mustafa considered two classes of single wave equation and proved general and explicit decay results of energy under a more general class of relaxation function satisfying

$$g'(t) \leq -\xi(t)H(g(t)).$$

For coupled wave system, Han and Wang [10] studied a coupled wave system with nonlinear weak dampings and finite memories. They proved local and global existence and finite time blow-up of solutions. A general decay result was established by Said-Houari et al. [29], and was extended by Messaoudi et al. [19] to wave system with past histories. Messaoudi and Tatar [20] considered a coupled system only with viscoelastic terms, and proved exponential decay and polynomial decay results, which was improved by Mustafa [22]. Recently, Al-Gharabli and Kafini considered the system in [20] and established a more general decay result by using some properties of convex functions, see [1].

The main question which can be asked here is the following: Whether can we get general and explicit decay rates for coupled Lamé system (1.1)–(1.4) under the different more general assumptions of different relaxations? Motivated by [6] and [25,26], in this paper, we intend to consider (1.1)–(1.4) with $g_i'(t) \leq -\xi_i(t)H_i(g_i(t))$, $i = 1, 2$, which is more general than the one in [6]. We establish explicit and general decay of system (1.1)–(1.4). Hence we extend the results of a single wave equation in [25, 26] to coupled wave equations. It must to be point out that the decay results established here are optimal exponential and polynomial rates for $1 \leq q < 2$ when $H(s) = s^q$, which improved the previous known results for $1 \leq q < \frac{3}{2}$. In addition, the energy decay result established in [6] is a special case of our result when the function $H(s)$ is linear. Since the decay result in the present work holds for $\lambda + \mu = 0$, our result also improves the ones in [1,20,22] and so on. Here the proof rely mainly on the construction of a Lyapunov functional. We adopt the idea of Mustafa [25,26] and Messaoudi and Hassan [21] and some properties of convex functions developed by Lasiecka and Tataru [12] and Alabau-Boussouira and Cannarsa [2].

The rest of this paper is as follows. In Section 2, we give some assumptions and our main results. In Section 3, we establish the general decay result of the energy.

2 Assumptions and main results

In the following, the constant $\delta > 0$ is the embedding constant

$$\delta\|u\|^2 \leq \|\nabla u\|^2, \quad \delta\|v\|^2 \leq \|\nabla v\|^2,$$

for $u \in H_0^1(\Omega)$. We write $\|\cdot\|$ instead of $\|\cdot\|_2$. The constant $c > 0$ denotes a generic constant.

We assume for $i = 1, 2$,

(A1) $g_i(t) : \mathbb{R}^+ \to \mathbb{R}^+$ are C^1 functions, which are increasing, satisfying

$$g_i(0) > 0 \quad \text{and} \quad \mu - \int_0^\infty g_i(s)ds = l_i > 0. \tag{2.1}$$

(A2) There exist two C^1 functions $H_i : \mathbb{R}^+ \to \mathbb{R}^+$ which are linear or are strictly increasing and strictly convex functions of class $C^2(\mathbb{R}^+)$ on $(0, r]$, $r \leq g_i(0)$, with $H_i(0) = H_i'(0) = 0$, such that

$$g_i'(t) \leq -\xi_i(t) H_i(g_i(t)), \quad \forall\, t \geq 0, \tag{2.2}$$

where $\xi_i(t)$ are C^1 functions satisfying

$$\xi_i(t) > 0, \quad \xi_i'(t) \leq 0, \qquad \forall\, t \geq 0.$$

Remark 1. It follows from (A1) that $\lim_{t\to+\infty} g_i(t) = 0$. We know that there exists some $t_1 \geq 0$ large enough such that

$$g_i(t_1) = r \Rightarrow g_i(t) \leq r, \quad \forall\, t \geq t_1.$$

For completeness, we give the existence of global solutions proved in [7].

Theorem 1. *Suppose* (2.1) *holds. If the initial data* $(u_0, v_0) \in [H^2(\Omega) \cap H_0^1(\Omega)]^2$, $(u_1, v_1) \in [L^2(\Omega)]^2$, *then problem* (1.1)–(1.4) *has a unique weak solution* (u, v) *satisfying that for any* $T > 0$,

$$u, v \in C([0,\infty); [H^2(\Omega) \cap H_0^1(\Omega)]^2), \quad u_t, v_t \in C([0,\infty); [L^2(\Omega)]^2).$$

The total energy of system (1.1)–(1.4) is defined by

$$\begin{aligned} E(t) \quad = \quad & \frac{1}{2}\Bigg[\|u_t(t)\|^2 + \|v_t(t)\|^2 + (\lambda+\mu)\|\mathrm{div} u(t)\|^2 + (\lambda+\mu)\|\mathrm{div} v(t)\|^2 \\ & + \left(\mu - \int_0^t g_1(s)ds\right)\|\nabla u(t)\|^2 + \left(\mu - \int_0^t g_2(s)ds\right)\|\nabla v(t)\|^2 \\ & + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\Bigg] + \alpha \int_\Omega u(t)v(t)dx, \end{aligned}$$

where

$$(g \circ \omega)(t) = \int_0^t g(t-s)\|\omega(t) - \omega(s)\|^2 ds.$$

We give the following stability result.

Theorem 2. *Suppose (A1) and (A2) hold. Let* $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, $(v_0, v_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$. *Then the energy* $E(t)$ *satisfies*

$$E(t) \leq k_2 H_4^{-1}\left(k_1 \int_0^t \xi(s)ds\right), \quad \forall\, t > 0, \tag{2.3}$$

where k_1, k_2 *are positive constants.*

$$H_4(t) = \int_t^r \frac{1}{sH_0(s)} ds, \quad H_0(t) = \min\{H_1'(t), H_2'(t)\},$$

and $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$.

Corollary 1. Let $H_i(s) = s^p (i = 1, 2)$, i.e.,

$$g_i'(t) \leq -\xi_i(t) g_i^p(t), \quad 1 \leq p < 2.$$

Assume (A1) and (A2) hold, then we have

$$E(t) \leq \begin{cases} k \exp\left(-k_1 \int_0^t \xi(s) ds\right), \ if \ p = 1, \\ \hat{k}\left(1 + \int_0^t \xi(s) ds\right)^{-1/(p-1)}, \ if \ 1 < p < 2, \end{cases} \tag{2.4}$$

where $k, \hat{k}$ and k_1 are positive constants.

We end this section by giving two examples to illustrate explicit formulas for the decay rates of the energy. One can find in [25, 26].

Example 1. If $g_1(t) = g_2(t) = e^{-t^q}$ with $0 < q < 1$, then we know that $g_i'(t) = -H_i(g_i(t))$ $(i = 1, 2)$ where $H_1(t) = H_2(t) = qt/[\ln(1/t)]^{\frac{1}{q}-1}$. Since

$$H_1'(t) = H_2'(t) = \frac{(1-q) + q \ln\left(\frac{1}{t}\right)}{\left[\ln\left(\frac{1}{t}\right)\right]^{\frac{1}{q}}}, \quad H_1''(t) = H_2''(t) = \frac{(1-q)\left[\ln\left(\frac{1}{t}\right) + \frac{1}{q}\right]}{t\left[\ln\left(\frac{1}{t}\right)\right]^{\frac{1}{q}+1}},$$

then the functions H_1 and H_2 satisfy (A2) on the interval $(0, r]$ for any $0 < r < 1$. Then we can get $E(t) \leq c_1 e^{-c_2 t^q}$.

Example 2. If $g_1(t) = g_2(t) = \dfrac{1}{(t+e)[\ln(t+e)]^p}$ with $p > 1$, then we have $g_1'(t) = g_2'(t) = -\dfrac{[\ln(t+e)+p]}{(t+e)^2[\ln(t+e)]^{p+1}}$. Clearly

$$g_i'(t) = -\frac{[\ln(t+e)+p]}{(t+e)\ln(t+e)}(g_i(t)).$$

We infer from $(2.4)_1$ that

$$E(t) \leq c_1 \exp\left(-c_2 \int_0^t \frac{[\ln(t+e)+p]}{(t+e)\ln(t+e)} ds\right) = \frac{c_1}{((t+e)[\ln(t+e)]^p)^{c_2}}.$$

As $c_2 \leq 1$, this is slower rate than $g_i(t)$. On the other hand,

$$g_i'(t) = -\frac{[\ln(t+e)+p]}{(t+e)^{1-\frac{1}{p}}}(g_i(t))^{1+\frac{1}{p}}.$$

By $(2.4)_2$, we get for large t

$$E(t) \leq c_3\left(1 + \int_0^t \frac{\ln(t+e)+p}{(t+e)^{1-\frac{1}{p}}} ds\right)^{-p} \leq \frac{c_3}{(t+e)[\ln(t+e)]^p}.$$

This is the same rate as $g_i(t)$.

3 Proof of Theorem 2

In this section, we will prove Theorem 2.

3.1 Technical lemmas

Lemma 1. *(**Energy identity**)([7]) The energy $E(t)$ satisfies that for any $t \geq 0$,*

$$\begin{aligned} E'(t) &= -\mu_1\|\nabla u_t(t)\|^2 - \mu_2\|\nabla v_t(t)\|^2 + \frac{1}{2}[(g_1' \circ \nabla u)(t) + (g_2' \circ \nabla v)(t)] \\ &\quad -\frac{1}{2}g_1(t)\|\nabla u(t)\|^2 - \frac{1}{2}g_2(t)\|\nabla v(t)\|^2 \leq 0. \end{aligned} \tag{3.1}$$

As in [25, 26], for any $0 < \zeta < 1$, we define

$$C_{\zeta,i} = \int_0^\infty \frac{g_i^2(s)}{\zeta g_i(s) - g_i'(s)} ds \quad and \quad h_i(t) = \zeta g_i(t) - g_i'(t), \quad i = 1, 2.$$

Lemma 2. *The functional $\phi(t)$*

$$\phi(t) = \int_\Omega u(t)u_t(t)dx + \int_\Omega v(t)v_t(t) + \frac{\mu_1}{2}\int_\Omega |\nabla u(t)|^2 dx + \frac{\mu_2}{2}\int_\Omega |\nabla v(t)|^2 dx,$$

satisfies for any $t \geq 0$,

$$\begin{aligned} \phi'(t) &\leq -\frac{l_1}{2}\|\nabla u(t)\|^2 - \frac{l_2}{2}\|\nabla v(t)\|^2 + \|u_t(t)\|^2 + \|v_t(t)\|^2 \\ &\quad -(\lambda+\mu)\|divu(t)\|^2 - (\lambda+\mu)\|divv(t)\|^2 + \frac{C_{\zeta,1}}{2l_1}(h_1 \circ \nabla u)(t) \\ &\quad +\frac{C_{\zeta,2}}{2l_2}(h_2 \circ \nabla u)(t) - 2\alpha\int_\Omega u(t)v(t)dx. \end{aligned} \tag{3.2}$$

Proof. From (1.1) we infer that

$$\begin{aligned} \phi'(t) =& \|u_t\|^2 + \|v_t\|^2 - \left(\mu - \int_0^t g_1(s)ds\right)\|\nabla u\|^2 - \left(\mu - \int_0^t g_2(s)ds\right)\|\nabla v\|^2 \\ & -(\lambda+\mu)\|\mathrm{div}u\|^2 - (\lambda+\mu)\|\mathrm{div}v\|^2 - 2\alpha\int_\Omega uvdx \\ & +\int_\Omega \nabla u(t)\int_0^t g_1(t-s)(\nabla u(s) - \nabla u(t))dsdx \\ & +\int_\Omega \nabla v(t)\int_0^t g_2(t-s)(\nabla v(s) - \nabla v(t))dsdx. \end{aligned} \tag{3.3}$$

Hölder's inequality gives us

$$\int_\Omega \left(\int_0^t g_1(t-s)|\nabla u(s) - \nabla u(t)|ds\right)^2 dx$$

$$
\begin{aligned}
&= \int_\Omega \left(\int_0^t \frac{g_1(t-s)}{\sqrt{\zeta g_1(t-s) - g_1'(t-s)}} \right. \\
&\qquad \left. \times \sqrt{\zeta g_1(t-s) - g_1'(t-s)} |\nabla u(s) - \nabla u(t)| ds \right)^2 dx \\
&\le \left(\int_0^t \frac{g_1^2(s)}{\zeta g_1(s) - g_1'(s)} ds \right) \int_\Omega \int_0^t [\zeta g_1(t-s) - g_1'(t-s)] \\
&\qquad \times |\nabla u(s) - \nabla u(t)|^2 ds dx \le C_{\zeta,1}(h_1 \circ \nabla u).
\end{aligned}
\tag{3.4}
$$

By Young's inequality and (3.4), we deduce that

$$
\begin{aligned}
&\int_\Omega \nabla u(t) \int_0^t g_1(t-s)(\nabla u(s) - \nabla u(t)) ds dx \\
&\le \frac{l_1}{2} \|\nabla u\|^2 + \frac{1}{2l_1} \int_\Omega \left(\int_0^t g_1(t-s) |\nabla u(s) - \nabla u(t)| ds \right)^2 dx \\
&\le \frac{l_1}{2} \|\nabla u\|^2 + \frac{C_{\zeta,1}}{2l_1} (h_1 \circ \nabla u).
\end{aligned}
\tag{3.5}
$$

Similarly,

$$
\int_\Omega \nabla v(t) \int_0^t g_2(t-s)(\nabla v(s) - \nabla v(t)) ds dx \le \frac{l_2}{2} \|\nabla v\|^2 + \frac{C_{\zeta,2}}{2l_2} (h_2 \circ \nabla v). \tag{3.6}
$$

Replacing (3.5) and (3.6) in (3.3), we can get (3.2). $\square$

The same arguments as in [25, 26], we can get the following three lemmas.

Lemma 3. *The functional $\theta_1(t)$ defined by*

$$
\theta_1(t) = \int_\Omega \int_0^t \sigma_1(t-s) |\nabla u(s)|^2 ds dx,
$$

where $\sigma_1(t) = \int_t^\infty g_1(s) ds$, satisfies

$$
\theta_1'(t) \le -\frac{1}{2}(g_1 \circ \nabla u) + 3(\mu - l_1) \|\nabla u\|^2. \tag{3.7}
$$

Proof. Clearly $\sigma_1'(t) = -g_1(t)$. Then

$$
\begin{aligned}
\theta_1'(t) &= \sigma_1(0) \|\nabla u\|^2 - \int_\Omega \int_0^t g_1(t-s) |\nabla u(s)|^2 ds dx \\
&= -\int_\Omega \int_0^t g_1(t-s) |\nabla u(s) - \nabla u(t)|^2 ds dx + \sigma_1(t) \|\nabla u\|^2 \\
&\quad -2 \int_\Omega \nabla u(t) \int_0^t g_1(t-s)(\nabla u(s) - \nabla u(t)) ds dx.
\end{aligned}
$$

By using Young's inequality, we obtain

$$
\begin{aligned}
&-2\int_\Omega \nabla u(t)\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx\\
&\qquad\le 2(\mu-l_1)\|\nabla u\|^2+\frac{\int_0^t g_1(s)ds}{2(\mu-l_1)}(g_1\circ\nabla u).
\end{aligned}
$$

Since $\sigma_1(t)\le\sigma_1(0)=\mu-l_1$ and $\displaystyle\int_0^t g_1(s)ds\le\mu-l_1$, we can obtain (3.7). □

The same arguments as in Lemma 3, we can get the following lemma.

Lemma 4. *The functional $\theta_2(t)$ defined by*

$$
\theta_2(t)=\int_\Omega\int_0^t \sigma_2(t-s)|\nabla v(s)|^2dsdx,
$$

where $\sigma_2(t)=\displaystyle\int_t^\infty g_2(s)ds$, satisfies

$$
\theta_2'(t)\le-\frac{1}{2}(g_2\circ\nabla v)+3(\mu-l_2)\|\nabla v\|^2. \tag{3.8}
$$

Now we define the functional $F(t)$

$$
F(t):=NE(t)+N_1\phi(t),
$$

where N and N_1 are positive constants. It is easy to get that for N large, there exist $\beta_1>0$ and $\beta_2>0$ such that

$$
\beta_1E(t)\le F(t)\le\beta_2E(t).
$$

Lemma 5. *It holds that for any $t\ge0$,*

$$
\begin{aligned}
F'(t)\quad\le\quad &-4(\mu-l_1)\|\nabla u(t)\|^2-4(\mu-l_2)\|\nabla v(t)\|^2-\frac{\mu_1}{2}\|\nabla u_t(t)\|^2\\
&-\frac{\mu_2}{2}\|\nabla v_t(t)\|^2-c\|divu(t)\|^2-c\|divv(t)\|^2\\
&-c\int_\Omega u(t)v(t)dx+\frac{1}{4}(g_1\circ\nabla u)(t)+\frac{1}{4}(g_2\circ\nabla v)(t).
\end{aligned} \tag{3.9}
$$

Proof. Combining (3.1)–(3.2), and noting $g_i'=\zeta g_i-h_i$ $(i=1,2)$, we can infer that for any $t>0$,

$$
\begin{aligned}
F'(t)\quad\le\quad &-\left(\mu_1N-\frac{N_1}{\delta}\right)\|\nabla u_t\|^2-\left(\mu_2N-\frac{N_1}{\delta}\right)\|\nabla v_t\|^2-\frac{l_1}{2}N_1\|\nabla u\|^2\\
&-\frac{l_2}{2}N_1\|\nabla v\|^2-(\lambda+\mu)N_1\|\mathrm{div}u\|^2-(\lambda+\mu)N_1\|\mathrm{div}v\|^2\\
&+\frac{N}{2}\zeta(g_1\circ\nabla u)+\frac{N}{2}\zeta(g_2\circ\nabla v)-2\alpha N_1\int_\Omega u(t)v(t)dx\\
&-\left(\frac{N}{2}-\frac{C_{\zeta,1}}{2l_1}N_1\right)(h_1\circ\nabla u)-\left(\frac{N}{2}-\frac{C_{\zeta,2}}{2l_2}N_1\right)(h_2\circ\nabla v),
\end{aligned}
$$

where we used Poincaré's inequalities $\delta\|u_t\|^2 \le \|\nabla u_t\|^2$ and $\delta\|v_t\|^2 \le \|\nabla v_t\|^2$. First of all we choose N_1 large so that

$$\frac{l_1}{2}N_1 > 4(\mu - l_1), \quad \frac{l_2}{2}N_1 > 4(\mu - l_2).$$

Note that

$$0 < \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i'(s)} < \frac{\zeta g_i^2(s)}{-g_i'(s)}, \quad i = 1, 2.$$

Then for any $s \in [0, \infty)$, we get

$$\lim_{\zeta \to 0} \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i'(s)} = 0, \quad i = 1, 2.$$

By using the fact $\frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i'(s)} < g_i(s)$ $(i = 1, 2)$, we can get

$$\lim_{\zeta \to 0} \zeta C_{\zeta,i} = \lim_{\zeta \to 0} \int_0^\infty \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i'(s)} ds = 0, \quad i = 1, 2.$$

Thus there exist some ζ_0 $(0 < \zeta_0 < 1)$ such that if $\zeta < \zeta_0$ then

$$\zeta C_{\zeta,i} < l_i / N_1, \quad i = 1, 2.$$

At last, for any fixed N_1, we choose N large enough and choose ζ satisfying

$$N > \frac{1}{2} + \frac{N_1}{\delta \mu_i}, \quad \zeta = \frac{1}{N} < \zeta_0, \quad i = 1, 2.$$

Then we have

$$\mu_i N - \frac{N_1}{\delta} > \frac{\mu_i}{2}, \quad \frac{N}{2} - \frac{C_{\zeta,i}}{2l_i} N_1 > 0, \quad i = 1, 2.$$

□

3.2 Proof of Theorem 2

Taking into account (3.9), we can get that there exist some constant $m > 0$,

$$F'(t) \le -mE(t) + c(g_1 \circ \nabla u)(t) + c(g_2 \circ \nabla v)(t), \quad \forall\, t > 0. \tag{3.10}$$

Case 1. The function $H(t)$ is linear. We multiply (3.10) by $\xi(t)$ and use (2.1) and (3.1) to get

$$\begin{aligned} \xi(t)F'(t) &\le -m\xi(t)E(t) + c\xi(t)(g_1 \circ \nabla u)(t) + c\xi(t)(g_2 \circ \nabla v)(t) \\ &\le -m\xi(t)E(t) - cE'(t). \end{aligned} \tag{3.11}$$

Define $\mathcal{E}(t) = \xi(t)F(t) + cE(t)$. We know that $\mathcal{E}(t)$ is equivalent to $E(t)$. Noting that $\xi(t)$ is nonincreasing, then we obtain from (3.11) that for any $t \ge 0$,

$$\mathcal{E}'(t) \le -m\xi(t)E(t),$$

which gives us

$$E(t) \le c_1 \exp\Big(- c_2 \int_0^t \xi(s)ds\Big).$$

Case 2. The function $H(t)$ is nonlinear. Define $\mathcal{G}(t) = F(t) + \theta_1(t) + \theta_2(t)$. It follows from (3.7), (3.8) and (3.9) that there exist a positive constant b such that for any $t \ge 0$,

$$\begin{aligned} \mathcal{G}'(t) \quad &\le \quad -(\mu - l_1)\|\nabla u\|^2 - (\mu - l_2)\|\nabla v\|^2 - \frac{\mu_1}{2}\|\nabla u_t\|^2 - \frac{\mu_2}{2}\|\nabla v_t\|^2 \\ &\quad -c\|\mathrm{div} u\|^2 - c\|\mathrm{div} v\|^2 - \frac{1}{4}(g_1 \circ \nabla u) - \frac{1}{4}(g_2 \circ \nabla v) \le -bE(t) \le 0. \end{aligned}$$

Then

$$b\int_0^t E(s)ds \le \mathcal{G}(0) - \mathcal{G}(t) \le \mathcal{G}(0) \ \Rightarrow \ \int_0^\infty E(s)ds < \infty. \qquad (3.12)$$

Define

$$\begin{aligned} I_1(t) &= q\int_0^t \int_\Omega |\nabla u(t) - \nabla v(t-s)|^2 dxds, \\ I_2(t) &= q\int_0^t \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dxds. \end{aligned}$$

By (3.12), we can choose a constant $0 < q < 1$ so that

$$I_i(t) < 1, \quad i = 1, 2, \quad \forall\ t \ge 0. \qquad (3.13)$$

We assume that $I_i(t) > 0$ for all $t \ge 0$, or else (3.10) implies an exponential decay. We define $\lambda_1(t)$ and $\lambda_2(t)$ by

$$\begin{aligned} \lambda_1(t) &= -\int_0^t g_1'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds, \\ \lambda_2(t) &= -\int_0^t g_2'(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dxds. \end{aligned}$$

It is obvious that $\lambda_i(t) \le -cE'(t), \quad i = 1, 2$. Noting that $H_i(t)$ is strictly convex on $(0, r]$ and $H_i(0) = 0$, we have

$$H_i(\nu x) \le \nu H_i(x), \quad i = 1, 2,$$

provided $0 \le \nu \le 1$ and $x \in (0, r]$. By using (2.2), (3.13) and Jensen's inequality, we can obtain

$$\begin{aligned} \lambda_1(t) \quad &= \quad \frac{1}{qI_1(t)} \int_0^t I_1(t)(-g_1'(s)) \int_\Omega q|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\ge \quad \frac{1}{qI_1(t)} \int_0^t I_1(t)\xi_1(s)H_1(g_1(s)) \int_\Omega q|\nabla u(t) - \nabla u(t-s)|^2 dxds \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{qI_1(t)}\int_0^t I_1(t)\xi_1(s)H_1(g_1(s))\int_\Omega q|\nabla u(t)-\nabla u(t-s)|^2dxds\\
&\geq \frac{\xi_1(t)}{qI_1(t)}\int_0^t H_1(I_1(t)g_1(s))\int_\Omega q|\nabla u(t)-\nabla u(t-s)|^2dxds\\
&\geq \frac{\xi_1(t)}{q}H_1\left(\frac{1}{I_1(t)}\int_0^t I_1(t)g_1(s)\int_\Omega q|\nabla u(t)-\nabla u(t-s)|^2dxds\right)\\
&= \frac{\xi_1(t)}{q}H_1\left(q\int_0^t g_1(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2dxds\right)\\
&= \frac{\xi_1(t)}{q}\overline{H}_1\left(q\int_0^t g_1(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2dxds\right). \qquad (3.14)
\end{aligned}
$$

Here $\overline{H}_1$ is an extension of H_1, which is strictly convex and strictly increasing C^2 function on $(0,\infty)$. We have from (3.14) that

$$\int_0^t g_1(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2dxds \leq \frac{1}{q}\overline{H}_1^{-1}\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right).$$

Similarly, we have

$$\int_0^t g_2(s)\int_\Omega |\nabla v(t)-\nabla v(t-s)|^2dxds \leq \frac{1}{q}\overline{H}_2^{-1}\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right).$$

We infer from (3.10) that for any $t\geq 0$,

$$F'(t)\leq -mE(t)+c\overline{H}_1^{-1}\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right)+c\overline{H}_2^{-1}\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right). \qquad (3.15)$$

Let's denote

$$H_0(t)=\min\{\overline{H}_1', \overline{H}_2'\}.$$

For $\varepsilon_0<r$, we define the function $\mathcal{K}_1(t)$

$$\mathcal{K}_1(t)=H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)F(t)+E(t),$$

which is equivalent to $E(t)$. Since $E'(t)\leq 0$, $\overline{H}_i'>0$ and $\overline{H}_i''>0$, we obtain from (3.15) that

$$
\begin{aligned}
\mathcal{K}_1'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}H_0'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)F(t)+H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)F'(t)+E'(t)\\
&\leq -mE(t)H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{H}_1^{-1}\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right)\\
&\quad +cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{H}_2^{-1}\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right). \qquad (3.16)
\end{aligned}
$$

Now we denote the conjugate function of the convex function $\overline{H}_i$ by $\overline{H}_i^*$, see, for instance, Arnold [3]. Then

$$\overline{H}_i^*(s)=s(\overline{H}_i')^{-1}(s)-\overline{H}_i[(\overline{H}_i')^{-1}(s)],\quad i=1,2,$$

which satisfies Young's inequality,

$$AB_i \leq \overline{H}_i^*(A) + \overline{H}_i(B_i), \quad i = 1, 2.$$

Setting $A{=}H_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$, $B{=}\overline{H}_i^{-1}\left(\frac{q\lambda_i(t)}{\xi_i(t)}\right)$, and using $\overline{H}_i^*(s){\leq}s(\overline{H}_i')^{-1}(s)$ and (3.16), we conclude

$$\begin{aligned}
\mathcal{K}_1'(t) \quad \leq \quad & -mE(t)H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\overline{H}_1^*\left(H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) + c\frac{q\lambda_1(t)}{\xi_1(t)} \\
& + c\overline{H}_2^*\left(H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) + c\frac{q\lambda_2(t)}{\xi_2(t)} \\
\leq \quad & -mE(t)H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)(\overline{H}_1')^{-1}\left(H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) \\
& + c\frac{q\lambda_1(t)}{\xi_1(t)} + cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)(\overline{H}_2')^{-1}\left(H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) + c\frac{q\lambda_2(t)}{\xi_2(t)} \\
\leq \quad & -mE(t)H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)(\overline{H}_1')^{-1}\left(\overline{H}_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) \\
& + c\frac{q\lambda_1(t)}{\xi_1(t)} + cH_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)(\overline{H}_2')^{-1}\left(\overline{H}_2'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right) + c\frac{q\lambda_2(t)}{\xi_2(t)} \\
\leq \quad & -(mE(0) - c\varepsilon_0)\frac{E(t)}{E(0)}H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cq\left(\frac{\lambda_1(t)}{\xi_1(t)} + \frac{\lambda_2(t)}{\xi_2(t)}\right). \quad (3.17)
\end{aligned}$$

Multiplying (3.17) by $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$, we get

$$\begin{aligned}
\xi(t)\mathcal{K}_1'(t) \quad \leq \quad & -(mE(0) - c\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cq\left(\lambda_1(t) + \lambda_2(t)\right) \\
\leq \quad & -(mE(0) - c\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - cE'(t). \quad (3.18)
\end{aligned}$$

Define the functional $\mathcal{K}_2(t)$ by

$$\mathcal{K}_2(t) = \xi(t)\mathcal{K}_1(t) + cE(t).$$

We know that there exist two positive constants β_3 and β_4 such that

$$\beta_3\mathcal{K}_2(t) \leq E(t) \leq \beta_4\mathcal{K}_2(t). \quad (3.19)$$

Making a appropriate choice of ε_0, we infer from (3.18) that for some constant $k > 0$,

$$\mathcal{K}_2'(t) \leq -\zeta\xi(t)\frac{E(t)}{E(0)}H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) := -\zeta\xi(t)H_3\left(\frac{E(t)}{E(0)}\right), \quad (3.20)$$

where $H_3(t) = tH_0(\varepsilon_0 t)$.

From $0 \leq \varepsilon_0\frac{E(t)}{E(0)} < r$ we infer that for any $t > 0$

$$\begin{aligned}
H_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) \quad = \quad & \min\left\{\overline{H}_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right), \overline{H}_2'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right\} \\
= \quad & \min\left\{H_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right), H_2'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right\}.
\end{aligned}$$

Denote $R(t) = \dfrac{\beta_3 \mathcal{K}_2(t)}{E(0)}$. Using (3.19), we see that

$$R(t) \sim E(t). \tag{3.21}$$

Since $H_3'(t) = H_0(\varepsilon_0 t) + \varepsilon_0 t H_0'(\varepsilon_0 t)$, then, using the strict convexity of H_0 on $(0, r]$, we know that $H_0'(t), H_0(t) > 0$ on $(0, 1]$. By (3.20), we obtain that there exists a constant $k_1 > 0$ such that for all $t \geq t_1$,

$$R'(t) \leq -k_1 \xi(t) H_3(R(t)). \tag{3.22}$$

Integrating (3.22) over $(0, t)$, we arrive at

$$\int_0^t \frac{-R'(s)}{H_3(R(s))} ds \geq k_1 \int_0^t \xi(s) ds \Rightarrow \int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(0)} \frac{1}{s H_0(s)} ds \geq k_1 \int_0^t \xi(s) ds.$$

Since H_4, defined by

$$H_4(t) = \int_t^r \frac{1}{s H_0(s)} ds,$$

is strictly decreasing on $(0, r]$ and $\lim\limits_{t \to 0} H_4(t) = +\infty$, we find that

$$R(t) \leq \frac{1}{\varepsilon_0} H_4^{-1} \left(k_1 \int_0^t \xi(s) ds \right). \tag{3.23}$$

Then we can get (2.3) from (3.21) and (3.23). $\square$

Acknowledgements

The authors would like to thank the anonymous referees very much for their valuable suggestions which improved this paper. Baowei Feng was supported by the National Natural Science Foundation of China (No. 11701465) and by the Fundamental Research Funds for the Central Universities (No. JBK1902026). Haiyan Li was supported by the Scientific Research Funds of North Minzu University (No. 2018XYZSX02) and First-Class Disciplines Foundation of Ningxia (No. NXYLXK2017B09).

References

[1] M.M. Al-Gharabli and M. Kafini. A general decay result of a coupled system of nonlinear wave equations. *Rend. Circ. Mat. Palermo, II. Ser*, **67**(1):145–157, 2018. https://doi.org/10.1007/s12215-017-0301-2.

[2] F. Alabau-Boussouira and P. Cannarsa. A general method for proving sharp energy decay rates for memory-dissipative evolution equations. *C. R. Acad. Sci. Paris Ser. I*, **347**(15–16):867–872, 2009. https://doi.org/10.1016/j.crma.2009.05.011.

[3] V.I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, New York, 1989. https://doi.org/10.1007/978-1-4757-2063-1.

[4] A. Bchatnia and M. Daoulatli. Behavior of the energy for Lamé systems in bounded domains with nonlinear damping and external force. *Electron. J. Differential Equations*, **2013**:1–17, 2013.

[5] A. Bchatnia and A. Guesmia. Well-posedness and asymptotic stability for the Lamé system with infinite memories in a bounded domain. *Math. Control Relat. Fields*, **4**(4):451–463, 2014. https://doi.org/10.3934/mcrf.2014.4.451.

[6] A. Beniani, N. Taouaf and A. Benaissa. Well-posedness and exponential stability for coupled Lamé system with viscoelastic term and strong damping. *Computers Math. Appl.*, **75**(12):4397–4404, 2018. https://doi.org/10.1016/j.camwa.2018.03.037.

[7] A. Beniani, K. Zennir and A. Benaissa. Stability for the Lamé system with a time varying delay term in a nonlinear internal feedback. *Clifford Anal. Clifford Algebr. Appl.*, **5**:287–298, 2016.

[8] M.M. Cavalcanti, V.N.D. Cavalcanti, I. Lasiecka and F.A. Nascimento. Intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional dissipative effects. *Discrete Conti. Dyn. Syst. Ser. B*, **19**:1987–2012, 2014. https://doi.org/10.3934/dcdsb.2014.19.1987.

[9] M.M. Cavalcanti, V.N.D. Cavalcanti, I. Lasiecka and X. Wang. Existence and sharp decay rate estimates for a von Karman system with long memory. *Nonlinear Anal.: Real World Appl.*, **22**:289–306, 2015. https://doi.org/10.1016/j.nonrwa.2014.09.016.

[10] X. Han and M. Wang. General decay of energy for a viscoelastic equation with nonlinear damping. *Math. Methods Appl. Sci.*, **32**:346–358, 2009. https://doi.org/10.1002/mma.1041.

[11] I. Lasiecka, S.A. Messaoudi and M.I. Mustafa. Note on intrinsic decay rates for abstract wave equations with memory. *J. Math. Phys.*, **54**(3), 2013. https://doi.org/10.1063/1.4793988.

[12] I. Lasiecka and D. Tataru. Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping. *Differential Integral Equations*, **8**:507–533, 1993.

[13] I. Lasiecka and X. Wang. Intrinsic decay rate estimates for semilinear abstract second order equations with memory. *In New Prospects in Direct, Inverse and Control Problems for Evolution Equations, Springer INdAM Series*, **10**:271–303, 2014.

[14] W.J. Liu. General decay of solutions to a viscoelastic wave equation with nonlinear localized damping. *Ann. Acad. Sci. Fenn. Math.*, **34**:291–302, 2009.

[15] W.J. Liu. General decay rate estimate for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms. *J. Math. Phys.*, **50**:113506, 2009. https://doi.org/10.1063/1.3254323.

[16] S.A. Messaoud and M.I. Mustafa. On the control of solutions of viscoelastic equations with boundary feedback. *Nonlinear Anal.: Real World Appl.*, **10**(5):3132–3140, 2009. https://doi.org/10.1016/j.nonrwa.2008.10.026.

[17] S.A. Messaoudi. General decay of solutions of a viscoelastic equation. *J. Math. Anal. Appl.*, **341**(2):1457–1467, 2008. https://doi.org/10.1016/j.jmaa.2007.11.048.

[18] S.A. Messaoudi. General decay of the solution energy in a viscoelastic equation with a nonlinear source. *Nonlinear Anal.*, **69**(8):2589–2598, 2008. https://doi.org/10.1016/j.na.2007.08.035.

[19] S.A. Messaoudi and M.M. Al-Gharabli. A general decay result of a nonlinear system of wave equations with infinite memories. *Appl. Math. Compu.*, **259**(15):540–551, 2015. https://doi.org/10.1016/j.amc.2015.02.085.

[20] S.A. Messaoudi and N. e. Tatar. Uniform stabilization of solutions of a nonlinear system of viscoelastic equations. *Appl. Anal.*, **87**(3):247–263, 2008. https://doi.org/10.1080/00036810701668394.

[21] S.A. Messaoudi and J.H. Hassan. *On the general decay for a system of viscoelastic wave equations.* In: Dutta H., Koinac L., Srivastava H. (eds) Current Trends in Mathematical Analysis and Its Interdisciplinary Applications, 2019. https://doi.org/10.1007/978-1-4757-2063-1.

[22] M.I. Mustafa. Well posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations. *Nonlinear Anal. Real World Appl.*, **13**(1):452–463, 2012. https://doi.org/10.1016/j.nonrwa.2011.08.002.

[23] M.I. Mustafa. On the control of the wave equation by memory-type boundary condition. *Discrete Contin. Dyn. Syst. Ser. A.*, **35**(3):1179–1192, 2015. https://doi.org/10.3934/dcds.2015.35.1179.

[24] M.I. Mustafa. Uniform decay rates for viscoelastic dissipative systems. *J. Dyn. Control Syst.*, **22**(1):101–116, 2016. https://doi.org/10.1007/s10883-014-9256-1.

[25] M.I. Mustafa. General decay result for nonlinear viscoelastic equations. *J. Math. Anal. Appl.*, **457**(1):134–152, 2018. https://doi.org/10.1016/j.jmaa.2017.08.019.

[26] M.I. Mustafa. Optimal decay rates for the viscoelastic wave equation. *Math. Methods Appl. Sci.*, **41**:192–204, 2018. https://doi.org/10.1002/mma.4604.

[27] M.I. Mustafa and S.A. Messaoudi. General stability result for viscoelastic wave equations. *J. Math. Phys.*, **53**(5), 2012. https://doi.org/10.1063/1.4711830.

[28] J. Park and S. Park. General decay for quasilinear viscoelastic equations with nonlinear weak damping. *J. Math. Phys.*, **50**(8), 2009. https://doi.org/10.1063/1.3187780.

[29] B. Said-Houari, S.A. Messaoudi and A. Guesmia. General decay of solutions of a nonlinear system of viscoelastic wave equations. *NoDEA Nonlinear Differ. Equ. Appl.*, **18**:659–684, 2011. https://doi.org/10.1007/s00030-011-0112-7.

[30] T.J. Xiao and J.Liang. Coupled second order semilinear evolution equations indirectly damped via memory effects. *J. Differ. Equ.*, **254**(5):2128–2157, 2013. https://doi.org/10.1016/j.jde.2012.11.019.