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SOME ESTIMATES FOR A SPECIAL LINEAR DIFFERENCE PARABOLIC EQUATION

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ABSTRACT

The finite-difference scheme for a special linear parabolic equation is investigated. A priori estimates for such finite-difference scheme are derived in the difference analogues of norm on Banach function spaces V_2 and $W_2^{2,1}$.

1. INTRODUCTION

A.A. Amosov and A.A. Zlotnik [2,3] obtained some results on the following linear difference scheme (LDS,$k = 0, 1$):

$$\overline{\partial}_t(\alpha V) = \delta(\kappa\delta V) + \Phi, \delta^k V\big|_{i=0,n} = 0, V\big|^{j=0} = V^0. \tag{1}$$

The unknown function V is defined on the grid $\overline{\omega}^h \times \overline{\omega}^\tau$ when $k = 0$ and V is defined on the grid $\overline{\omega}^h_{1/2} \times \overline{\omega}^\tau$ when $k = 1$. We suppose that $\delta^k V^0\big|_{i=0,n} = 0$. Eq. (1) is defined on the grid $\omega^h \times \omega^\tau$ when $k = 0$ and on the grid $\omega^h_{1/2} \times \omega^\tau$ when $k = 1$. So, the grids on which functions α, κ, and Φ are defined are known. Let $\Phi = \overline{\partial}_t \Phi_a + \Phi_b + \delta\Phi_c$, where $\Phi_c\big|_{i=0,n} = 0$ when $k = 1$.

LEMMA 1.1. *Let* $N^{-1} \le \alpha, \kappa$, *and* $q_l, r_l \in [1, \infty], l = 1, 2, 3, 4$; *and* $(2q_1)^{-1} + r_1^{-1} \le 5/4$, $(2q_l)^{-1} + r_l^{-1} < 1, l = 2, 3, 4$. *The next statements are valid:*

a) *If* $||\overline{\partial}_t\kappa||_{\infty,1} \le N$ $(\kappa^0 = \kappa^1)$ *then*

$$\begin{aligned} &||V||_Q + ||I_\tau \delta V||_{2,\infty} \\ &\quad \le K(N)(||\alpha^0 V^0 - \Phi_a^0||^{(-1);k} + ||\Phi_a||_Q + ||\Phi_b||_{1,1} + ||\Phi_c||_{2,1}), \end{aligned}$$

where

$$||V||^{(-1);0} = \sup_{W:W_0=W_n=0} \frac{(V,W)_{\omega^h}}{||W||^{(1)}}, \quad ||V||^{(-1);1} = \sup_{W} \frac{(V,W)_{\omega^h_{1/2}}}{||W||^{(1)}};$$

b) *If* $||\alpha^0||_\infty \leq N$, $||\overline{\partial}_t\alpha||_{q_2,r_2} \leq N$ *(or* $||\overline{\partial}_t\alpha||_{\infty,1} \leq N$*) and* $\tau_{\max} \leq \tau^0(N)$ *or if* $||\alpha^0||_\infty \leq N$, $\overline{\partial}_t\alpha \geq 0$ *then*

$$||\sqrt{\tau}\partial_t V||^2_Q + ||V||_{V_2} \leq$$
$$\leq K(N)(||V^0|| + ||\varPhi^0_a|| + ||\tau^{-1/2}\varPhi_a||_Q + ||\varPhi_b||_{q_1,r_1} + ||\varPhi_c||_Q);$$

c) *If conditions b) are valid and* $\varPhi_a = 0$ *then*

$$||V||_C \leq K(N)(||V^0||_\infty + ||\varPhi_b||_{q_3,r_3} + ||\varPhi_c||_{2q_4,2r_4});$$

d) *If* $\alpha, \kappa \leq N$, $||\overline{\partial}_t\alpha||_Q \leq N$, $||\delta\kappa||_{2,\infty} \leq N$ *then*

$$||V||^{(2,1)} \leq K(N)(||V^0||^{(1)} + ||\varPhi||_Q + ||V||_{V_2}).$$

2. NOTATION

The notation and conventions adopted here are the same as that introduced by A.A. Amosov and A.A. Zlotnik [3]. In the domain $\overline{Q}$ we introduce grids $\overline{\overline{\omega}}^h = \{\widetilde{q}_{-1} \leq 0 = \widetilde{q}_0 < \widetilde{q}_1 < \cdots < \widetilde{q}_n = L \leq \widetilde{q}_{n+1}\}$, $\overline{\omega}^h = \overline{\overline{\omega}}^h \setminus \{\widetilde{q}_{-1}, \widetilde{q}_{n+1}\}$, $\widetilde{\omega}^h = \overline{\omega}^h \setminus \{\widetilde{q}_0\}$, $\omega^h = \overline{\omega}^h \setminus \{\widetilde{q}_0, \widetilde{q}_n\}$ with stepsizes $h_i = \widetilde{q}_i - \widetilde{q}_{i-1}, 1 \leq i \leq n, h_0 = h_{n+1} = 0$, the grids $\overline{\omega}^h_{1/2} = \{\widetilde{q}_{i+1/2} | \widetilde{q}_{i+1/2} = (\widetilde{q}_i + \widetilde{q}_{i+1})/2, \ -1 \leq i \leq n\}$, $\widetilde{\omega}^h_{1/2} = \overline{\omega}^h_{1/2} \setminus \{\widetilde{q}_{-1/2}\}$, $\omega^h_{1/2} = \overline{\omega}^h_{1/2} \setminus \{\widetilde{q}_{-1/2}, \widetilde{q}_{n+1/2}\}$ with stepsizes $h_{i+1/2} = (h_i + h_{i+1})/2$, $0 \leq i \leq n$, and $\overline{\omega}^\tau = \{t_j | 0 = t_0 < t_1, \cdots, t_{\overline{n}} = T\}$, $\omega^\tau = \overline{\omega}^\tau \setminus \{t_0\}$, $\overset{\circ}{\omega}{}^\tau = \omega^\tau \setminus \{t_1\}$ with stepsizes $\tau_j = t_j - t_{j-1}, \ \ 0 < j \leq \overline{n}$. We assume $h_{\max} = \max\limits_{1 \leq i \leq n} h_i$, $\tau_{\max} = \max\limits_{1 \leq j \leq \overline{n}} \tau_j$, $\tau_0 = 0$. Let the grid $\overline{\overline{\omega}}^h$ be a quasiuniform, i.e. $N^{-1} \leq h_{i+1}/h_i \leq N \ (0 < i < n)$. We consider grid functions $Z = Z^j = Z_i = Z^j_i = Z(\widetilde{q}_i, t_j)$, $Z = Z^j = Z_{i+1/2} = Z^j_{i+1/2} = Z(\widetilde{q}_{i+1/2}, t_j)$, and denote $Z \in \boldsymbol{R}(\overline{\omega}^h \times \overline{\omega}^\tau)$ and $Z \in \boldsymbol{R}(\overline{\omega}^h_{1/2} \times \overline{\omega}^\tau)$, respectively. For functions $Z_k, k = 1, 2, ...$ we use the notation $Z_k = Z_{k;i}$ or $Z_k = Z_{k;i+1/2}$.

Let the grid functions $U \in \boldsymbol{R}(\overline{\omega}^h)$, $\widetilde{U} \in \boldsymbol{R}(\widetilde{\omega}^h)$, $\overset{\circ}{U} \in \boldsymbol{R}(\omega^h)$, $\overline{V} \in \boldsymbol{R}(\overline{\omega}^h_{1/2})$, $\widetilde{V} \in \boldsymbol{R}(\widetilde{\omega}^h_{1/2})$, $V \in \boldsymbol{R}(\omega^h_{1/2})$, $Y \in \boldsymbol{R}(\overline{\omega}^\tau)$, $\overset{\circ}{Y} \in \boldsymbol{R}(\omega^\tau)$ be determined on the grids $\overline{\omega}^h$, $\widetilde{\omega}^h$, ω^h, $\overline{\omega}^h_{1/2}$, $\widetilde{\omega}^h_{1/2}$, $\omega^h_{1/2}$, $\overline{\omega}^\tau$, ω^τ, respectively, and let Z be one of the functions U, $\widetilde{U}$, $\overset{\circ}{U}$, $\overline{V}$, $\widetilde{V}$, and V. In this section $\dot{U} \in R(\overline{\omega}^h)$ is zero continuation function of U, and $\overset{*}{Y} \in R(\overset{\circ}{\omega}{}^\tau)$ is zero continuation function of

$\overset{\circ}{Y}$, and $\overset{*}{U} \in R(\overline{\omega}^h)$ is zero continuation function of $\widetilde{U}$, and $\ddot{V} \in R(\overline{\omega}^h_{1/2})$ is even continuation function ($\ddot{V}_{-1/2} = V_{1/2}$, $\ddot{V}_{n+1/2} = V_{n-1/2}$) of V.

We introduce the following grid operators. For the functions $Z_i = Z(\widetilde{q}_i)$ defined on the grids $\overline{\omega}^h$ or $\overline{\omega}^h_{1/2}$ we assume

$$\delta Z_{i+1/2} = (Z_{i+1} - Z_i)/h_{i+1}, \quad sZ_{i+1/2} = (Z_i + Z_{i+1})/2, \quad Z\pm_{,i} = Z_{i\pm 1/2}$$

(i is integer or semi-integer indice). Let $\delta^k, s^k,\ k = 0,1,2$ be powers of operators δ, s. It is not difficult to prove that $\delta(Z_1Z_2) = \delta(Z_1)s(Z_2) + s(Z_1)\delta(Z_2)$ and

$$|s\overline{V}| \le 2s|\overline{V}|, \quad |sU| \le 2s|U|, \quad |\delta sU| \le 2s|\delta U|, \tag{2}$$

$$|\delta s\overline{V}| \le K(N)s|\delta\overline{V}|,\ |\delta s\ddot{V}| \le K(N)s|\overline{\partial}_t\dot{\delta V}|. \tag{3}$$

For the functions $Y^j = Y(t_j) \in \boldsymbol{R}(\overline{\omega}^\tau)$ we assume

$$\overset{\vee}{Y}{}^j = Y^{j-1}, 1 \le j \le \overline{n}, \quad \widehat{Y}^k = Y^{k+1}, 0 \le k < \overline{n},\ \overset{\vee}{Y}{}^0 = Y^0,\ \overline{\partial}_t Y^0 = 0$$

$$\overline{\partial}_t Y = (Y - \overset{\vee}{Y})/\tau, \quad \partial_t Y = (\widehat{Y} - Y)/\tau, \quad s_t Y = (Y + \overset{\vee}{Y})/2, \quad I^0_\tau Y = 0,$$

$$I^{k,j}_\tau Y = \sum_{k<l\le j} Y^l \tau_l,\ \overset{\circ}{I}{}^j_\tau Y = I^{1,j}_\tau Y,\ I^j Y = I^{0,j}_\tau Y.$$

Then the following formulae are valid:

$$\overline{\partial}_t(Y_1Y_2) = (\overline{\partial}_t Y_1)Y_2 + \overset{\vee}{Y}_1(\overline{\partial}_t Y_2),$$

$$I_\tau(Y_1Y_2) = Y_1 I_\tau Y_2 - I_\tau(\overline{\partial}_t Y_1\ \overset{\vee}{I}_\tau Y_2),$$

$$I^{k,j}_\tau(Y_1\overline{\partial}_t Y_2) = Y^j_1 Y^j_2 - Y^k_1 Y^k_2 - I^{k,j}_\tau(\overline{\partial}_t Y_1 Y_2).$$

From the norm $L_q(\Omega)$, $L_q(\Omega)$, $L_r(0,T)$ for $q, r \in [1,\infty)$ we get the norms $||\cdot||_{q,\overline{\omega}^h}$, $||\cdot||_{q,\omega^h_{1/2}}$, $||\cdot||_{r,\omega^\tau}$ if trapezoidal, midvalue, right rectangular integration rules are used, respectively. We introduce the following norms (ω is one of the grids on $[0, L]$)

$$||U||_{q,\omega^h} = ||\dot{U}||_{q,\overline{\omega}^h},\ ||\widetilde{U}||_{q,\omega^h} = ||\overset{*}{U}||_{q,\overline{\omega}^h},\ ||Y||_{\infty,\omega^\tau} = \max_{1\le j\le\overline{n}} |Y(t_j)|,$$

$$||Z||_\infty = \max_{x\in\omega} |Z(x)|,\ ||\cdot||_{q,r,\omega\times\omega^\tau} = ||\ ||\cdot||_{q,\omega}||_{r,\omega^\tau}, q, r \in [1,\infty],$$

as well as the inner product $(Z,\overline{Z})_\omega$ such that $(Z,Z)_\omega = ||Z||^2_{2,\omega}$. Then the formulae of the summation by parts

$$(U,\delta V)_{\omega^h} + (\delta U, V)_{\omega^h_{1/2}} = U_n V_{n-1/2} - U_0 V_{1/2}$$

$$(U,\delta\widetilde{V})_{\widetilde{\omega}^h}+(\delta U,\widetilde{V})_{\omega^h_{1/2}}=U_n\widetilde{V}_{n+1/2}-U_0\widetilde{V}_{1/2}$$

hold. We denote

$$||Z||_{Q^l,\omega}=(\ I^l_\tau||Z||^2_{2,\omega})^{1/2},\quad ||Z||_{2,\infty,Q^l,\omega}=\max\nolimits_{0\le j\le l}||Z^j||_{2,\omega},$$
$$||Z||_{C(Q^l),\omega}=\max\nolimits_{0\le j\le l}||Z^j||_{\infty,\omega},\quad 0<l\le\overline{n}.$$

Further on we omit subscripts ω, ω^τ, and $\omega\times\omega^\tau$ denoting a domain of grid functions. We use norms

$$||Z||_{V_2(Q^l)}=||Z||_{2,\infty,Q^l}+||\delta Z||_{Q^l},$$
$$||Z||^{(2,1)}_{Q^l}=||Z||_{V_2(Q^l)}+||\overline{\partial}_tZ||_{Q^l}+||\delta Z||_{V_2(Q^l)}.$$

Denote $||\cdot||_C=||\cdot||_{C(Q)}$, $||\cdot||_{V_2}=||\cdot||_{V_2(Q)}$, $||\cdot||^{(2,1)}=||\cdot||^{(2,1)}_Q$, $Q=Q^{\overline{n}}$, $||\cdot||=||\cdot||_2$, $||\cdot||^{(1)}=||\cdot||+||\delta\cdot||$.

It is not difficult to prove that $(0<\varepsilon\le 1,p,q,q_0,q_1,r,r_0,r_1\in[1,\infty])$

$$||\ sU\ ||_p\le||\ U\ ||_p,\ ||\ sV\ ||_p\le K(N)||\ V\ ||_p,$$
$$||Z||_\infty\le\varepsilon||\delta Z||+K_\varepsilon(N)||Z||_1,$$
$$||Z||_\infty+||\delta Z||_\infty\le\varepsilon||\delta^2Z||+K_\varepsilon(N)||Z||_1,$$
$$||Z||_q\le K(N)(||\delta Z||+||Z||)^{2/r}||Z||^{1-2/r},$$
$$|(a,Z)|\le\varepsilon||\delta Z||^2+K_\varepsilon(N)(||a||^{r_1}_{q_1}||Z||^{2-r_1}+||a||_{q_1}||Z||),$$
$$|\ (aZ_1,Z)\ |\le\varepsilon||\delta Z||^2+K_\varepsilon(N)\big(\ ||Z||^2+||a||^{r_1}_{q_1}(||Z||^2+||Z_1||^2_\infty)\big),$$
$$|\ (aZ_1,Z)\ |\le\varepsilon||\delta Z||^2+\varepsilon||\delta Z_1||^2+K_\varepsilon(N)\big(\ (1+||a||^{r_0}_{q_0})\ (||Z||^2+||Z_1||^2)\big),$$
$$(2q)^{-1}+r^{-1}=1/4,\ (2q_0)^{-1}+r_0^{-1}=1,\ (2q_1)^{-1}+r_1^{-1}=5/4.$$

LEMMA 2.1 (Difference Gronwall's L., see [1,3]). *Let functions $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, $B^{(1)}$, $B^{(2)}$, F be defined on the grid ω^τ. If function $Y\ge 0$ is defined on the grid $\overline{\omega}^\tau$ and satisfies inequality*

$$Y\le\overline{Y}^0+I_\tau(A^{(1)}Y+A^{(2)}\overset{\vee}{Y}+B^{(1)}Y^{1/2}+B^{(2)}\overset{\vee}{Y}{}^{1/2}+F)+\overset{\vee}{I}_\tau(A^{(3)}Y)$$

with $\overline{Y}^0=const\ge Y^0\ge 0$ and $\tau_jA^{(1),j}\le 1/2,\ 1\le j\le\overline{n}$, then the estimate

$$||Y||^{1/2}_\infty\le\big((\overline{Y}^0+||F||_1)^{1/2}+||B||_1\exp(||A||_1)\big)\exp(||A||_1)$$

is valid. There $A=|A^{(1)}|+|A^{(2)}|+|A^{(3)}|+,A^{(3)}|_{j=\overline{n}}=0,B=|B^{(1)}|+|B^{(2)}|,\|\cdot\|r=\|\cdot\|r,w^\tau$.

3. SPECIAL LINEAR PARABOLIC EQUATION

Now we consider another linear difference scheme (LDS1) [4]:

$$\overline{\partial}_t(\alpha V) = \beta\delta\Pi + \Phi, \quad V\big|_{i=0} = \Pi\big|_{i=n+1/2} = 0, \quad V\big|^{j=0} = V^0. \tag{4}$$

The unknown function V is defined on the grid $\overline{\omega}^h \times \overline{\omega}^\tau$ and $V^0\big|_{i=0} = 0$, Π is defined on the grid $\widetilde{\omega}^h_{1/2} \times \omega^\tau$. If we suppose that $P^0 = P^1$, $\gamma^0 = \gamma^1$, $\kappa^0 = \kappa^1$ then $\Pi = \Pi(V) = \Pi(V,\kappa,\gamma,P) = \kappa\delta V + \gamma s V - P$ is defined on the grid $\omega^h_{1/2} \times \overline{\omega}^\tau$. Eq. (4) is defined on the grid $\widetilde{\omega}^h \times \omega^\tau$. We suppose $\Phi_c|_{i=n+1/2} = 0$. Let $L = L(Z) = L(Z,\beta,\kappa,\gamma,P) = \beta\delta\Pi$, $\Lambda = \Lambda(Z) = \Lambda(Z,\beta,\kappa,\gamma,P,\Phi) = L(Z) + \Phi$. We denote $\Pi_l(V) = \Pi(V,\kappa_l,\gamma_l,P_l)$, $\Lambda_l(Z) = \Lambda_l(Z,\beta_l,\kappa_l,\gamma_l,P_l,\Phi_l)$, $l = 1,2$; and $\vec{\kappa} = (\kappa_1,\kappa_2)$, $\vec{\gamma} = (\gamma_1,\gamma_2)$, $\vec{P} = (P_1,P_2)$, $\vec{\Phi} = (\Phi_1,\Phi_2)$.

LEMMA 3.1. *Let* $N^{-1} \le \kappa_1,\kappa_2 \le N$, *and* $q_l \in [1,\infty], r_l \in [1,2], l = 1,2,3,4$, *and* $(2q_l)^{-1} + r_l^{-1} = 1$.

a) *Then*

$$K(N)^{-1}||\delta V||^2 \le (\Pi_1,\Pi_2) + K(N)||\vec{P}||^2 + K(N)(1 + ||\vec{\gamma}||^{2r_1}_{2q_1})||V||^2,$$
$$(\Pi_1,\Pi_2) \le K(N)||\delta V||^2 + K(N)||\vec{P}||^2 + K(N)(1 + ||\vec{\gamma}||^{2r_1}_{2q_1})||V||^2.$$

b) *If* $N^{-1} \le \beta_1,\beta_2 \le N$ *then*

$$\begin{aligned} K(N)^{-1}||\delta^2 V||^2 &\le (\Lambda_1,\Lambda_2) + K(N)d_*^2, \\ (\Lambda_1,\Lambda_2) &\le K(N)(||\delta^2 V||^2 + d_*^2), \end{aligned}$$

where

$$\begin{aligned} d_*^2 &= \left(||\vec{P}||^{(1)}\right)^2 + ||\vec{\Phi}||^2 + (1 + ||\delta\vec{\gamma}||^{2r_2}_{2q_2})||V||^2 \\ &\quad + (1 + ||\vec{\gamma}||^{2r_3}_{2q_3} + ||\delta\vec{\kappa}||^{2r_4}_{2q_4})||\delta V||^2. \end{aligned}$$

P r o o f. a) As $\Pi = \kappa\delta V + \gamma s V - P$ and $||\gamma s V - P||^2 \le K||P||^2 + K(\,|\gamma|^2, sVsV\,) \le \varepsilon||\delta V||^2 + K||P||^2 + K(1 + ||\gamma||^{2r}_{2q})||V||^2$ then we can consider only the case $\gamma_1 = \gamma_2 = 0$. In this case we have

$$K^{-1}||\delta V||^2 \le (\kappa_1\delta V, \kappa_2\delta V) \le K||\delta V||^2, \quad |\,(-P_1,-P_2)\,| \le K||\vec{P}||^2,$$
$$|\,(\kappa_i\delta V, -P_j)\,| \le \varepsilon||\delta V||^2 + K||\vec{P}||^2, i,j = 1,2, i \ne j.$$

So the part a) is proved.

b) As $||\gamma s V - P||^2 \le \varepsilon||\delta^2 V||^2 + K||P||^2 + K(1 + ||\gamma||^{2r}_{2q})||V||^2$ and

$$\begin{aligned} &||\delta\gamma ssV + s\gamma\delta sV - \delta P||^2 \\ &\le \varepsilon||\delta^2 V||^2 + K(1 + ||\delta\gamma||^{2r}_{2q}))||V||^2 + K(1 + ||\gamma||^{2r}_{2q})||\delta V|| + K||\delta P||^2 \end{aligned}$$

then we can consider only the case $\gamma_1 = \gamma_2 = 0$. We have $\Lambda = \beta s\kappa\delta^2 V + \beta\delta\kappa s\delta V - \beta\delta P + \Phi$ and

$$||\beta\delta\kappa s\delta V - \beta\delta P + \Phi||^2 \\ \leq \varepsilon||\delta^2 V||^2 + K\big(||\Phi||^2 + ||\delta P||^2 + (1 + ||\delta\kappa||_{2q}^{2r})||\delta V||^2\big).$$

So we must consider only case $\Lambda = \beta\delta^2 V + \Phi$. Then the proof of case b) follows from inequalities

$$K^{-1}||\delta^2 V||^2 \leq (\beta_1\delta^2 V, \beta_2\delta^2 V) \leq K||\delta^2 V||^2, |\ (\Phi_1, \Phi_2)\ | \leq K||\vec{\Phi}||^2, \\ |\ (\beta_i\delta^2 V, \Phi_j)\ | \leq \varepsilon||\delta^2 V||^2 + K||\vec{\Phi}_j||^2, i, j = 1, 2, i \neq j.$$

We consider equation (4) and denote $||Z||_Q^{(-1)} = \big|\big|\ ||Z||^{(-1);1}\big|\big|_{2,\omega^\tau}$, $Z^- = \min(Z, 0)$.

LEMMA 3.2. *Let* $N^{-1} \leq \kappa, \beta \leq N$, $N^{-1} \leq \alpha$, *and* $q_l, r_l \in [1, \infty]$, $(2q_l)^{-1} + r_l^{-1} \leq 1, l = 1, ..., 6$, $(2q_l)^{-1} + r_l^{-1} \leq 5/4, l = 7, 8$.

a) *If* $||\alpha^0||_\infty \leq N$, $||(\overline{\partial}_t\alpha)^-||_{q_1,r_1} \leq N$, $||\gamma||_{2q_2,2r_2} \leq N$, $||\delta\beta||_{2q_3,2r_3} \leq N$ *and* $\tau_{\max} \leq \tau^0(N)$, *then*

$$||\sqrt{\tau}\partial_t V||_Q^2 + ||V||_{V_2} + ||\Pi||_Q \leq K(N)(||V^0|| + ||\Phi_a^0|| \\ + ||\tau^{-1/2}\Phi_a||_Q + ||\Phi_b||_{q_7,r_7} + ||\Phi_c||_Q + ||P||_Q + ||V||_Q^2);$$

b) *If* $\alpha \leq N$, $||\overline{\partial}_t\alpha||_Q \leq N$, $||\partial_t\kappa||_{q_4,r_4} \leq N$, $||\delta\kappa||_{2q_5,2r_5} \leq N$, $||\gamma||_{2,\infty} \leq N$, $||\delta\gamma||_{2q_6,2r_6} \leq N$, $||\partial_t\gamma||_{q_8,r_8} \leq N$ *and* $\tau_{\max} \leq \tau^0(N)$, *then*

$$||V||^{(2,1)} + ||\sqrt{\tau}\partial_t\delta V||_Q + ||\delta\Pi||_Q \leq K(N)(||V^0||^{(1)} + ||\Phi||_Q \\ + ||P||_{V_2(Q)} + ||\overline{\partial}_t P||_Q^{(-1)} + ||\sqrt{\tau}\partial_t P||_Q + ||V||_{V_2}).$$

If $q_1 = q_2 = q_3 = 1, r_1 = r_2 = r_3 = \infty$ *in the case a) or* $q_5 = 1, r_5 = \infty$ *in the case b) then we can omit condition* $\tau_{\max} \leq \tau^0(N)$.

P r o o f. In this proof let conditions for q_l, r_l be equalities because $||\cdot||_{q,r} \leq K||\cdot||_{\overline{q},\overline{r}}$ if $q \leq \overline{q}$ and $r \leq \overline{r}$.

a) Let $d = ||V^0|| + ||\Phi_a^0|| + ||\tau^{-1/2}\Phi_a||_Q + ||\Phi_b||_{q_1,r_1} + ||\Phi_c||_Q + ||P||_Q$. We take the inner product of equation (4) with V, apply operator I_τ^l (we omit index l below in the proof) and get

$$(\overline{\partial}_t\alpha, V^2)_Q + 0.5||\sqrt{\tau}\overset{\vee}{\alpha}{}^{1/2}\overline{\partial}_t V||_Q^2 + 0.5||\sqrt{\alpha}V||^2 \\ = 0.5||\sqrt{\alpha^0}V^0||^2 + (\delta\Pi, \beta V)_Q + (\Phi_b, V)_Q + (\delta\Phi_c, V)_Q \\ + (\Phi_a, V) - (\Phi_a^0, V^0) - (\tau^{-1/2}\overset{\vee}{\Phi}_a, \sqrt{\tau}\partial_t V)_Q.$$

If $\overline{\partial}_t\alpha \geq 0$ we bound the first summand by zero, else we estimate $(-\overline{\partial}_t\alpha, V^2)_Q \leq \varepsilon||\delta V||_Q^2 + KI_\tau\big((1+||(\overline{\partial}_t\alpha)^-||_{q_1}^{r_1})||V||^2\big)$. Using the formula of summation by parts we get $(\delta\varPhi_c, V)_Q = -(\varPhi_c, \delta V)_Q$ and

$$(\delta\varPi, \beta V)_Q = -(\varPi, \delta\beta s V)_Q - (\kappa s\beta\delta V, \delta V)_Q + (s\beta P, \delta V)_Q - (\gamma s V, \delta V)_Q.$$

We estimate

$$\begin{aligned}
&|\ (\varPi, \delta\beta s V)_Q\ | \leq \varepsilon||\varPi||_Q^2 + \varepsilon||\delta V||_Q^2 + KI_\tau\big((1+||\delta\beta||_{2q_3}^{2r_3})||V||^2\big),\\
&|\ (\gamma s V, \delta V)_Q\ | \leq \varepsilon||\delta V||_Q^2 + KI_\tau\big((1+||\gamma||_{2q_2}^{2r_2})||V||^2\big),\\
&|\ (\varPhi_b, V)_Q\ | \leq \varepsilon||\delta V||_Q^2 + \varepsilon||V||^2 + K\,\overset{\vee}{I}_\tau\big((||\varPhi_b||_{q_7}^{r_7} + ||\varPhi_b||_{q_7})||V||\big) + Kd^2.
\end{aligned}$$

From lemma 3.1a and obtained estimates we get the inequality

$$\begin{aligned}
||V||^2 \leq KI_\tau\big(\ (||(\overline{\partial}_t\alpha)^-||_{q_1}^{r_1} + ||\gamma||_{2q_2}^{2r_2} + ||\delta\beta||_{2q_3}^{2r_3})||V||^2\ \big)\\
+K\,\overset{\vee}{I}_\tau\big((||\varPhi_b||_{q_7}^{r_7} + ||\varPhi_b||_{q_7})||V||\big) + Kd^2.
\end{aligned}$$

Now we use Gronwall's lemma which proves this part of the lemma. If $||(\overline{\partial}_t\alpha)^-||_{1,\infty} + ||\gamma||_{2,\infty} + ||\delta\beta||_{2,\infty} \leq N$ then $I_\tau\big((\cdot)||V||^2\big) \leq Kd^2$.

b) Let $d = ||V^0||^{(1)} + ||\varPhi||_Q + ||P||_{V_2(Q)} + ||\overline{\partial}_t P||_Q^{(-1)} + ||\sqrt{\tau}\overline{\partial}_t P||_Q + ||V||_{V_2}$. We consider the case $\alpha = 1$ because the general case we get with $\varPhi' = \varPhi\overset{\vee}{\alpha}{}^{-1} - \overline{\partial}_t\alpha\overset{\vee}{\alpha}{}^{-1}V$. We take inner product of equation (4) with $(\beta^{-1}\overline{\partial}_t V - \delta\varPi)/2$, apply the operator I_τ^l which yields

$$K^{-1}(\ ||\overline{\partial}_t V||_Q^2 + ||\delta\varPi||_Q^2) \leq (\ \overline{\partial}_t V, \delta\varPi\ )_Q + Kd^2.$$

Using the formula of summation by parts we have

$$(\overline{\partial}_t V, \delta\varPi)_Q = -(\overline{\partial}_t\delta V, \kappa\delta V)_Q + (\overline{\partial}_t\delta V, P)_Q + (\overline{\partial}_t\delta V, \gamma s V)_Q.$$

Each summand on the righthand side of this equality we rewrite in the form

$$\begin{aligned}
-0.5(\kappa, \tau(\overline{\partial}_t\delta V)^2)_Q - 0.5||\sqrt{\kappa}\delta V||^2 + 0.5||\sqrt{\kappa^0}\delta V^0||^2 + 0.5\,\overset{\vee}{I}_\tau(\partial_t\kappa, |\delta V|^2),\\
(P, \delta V) - (P^0, \delta V^0) - I_\tau(\overline{\partial}_t P, \delta V) + I_\tau(\tau\overline{\partial}_t P, \overline{\partial}_t\delta V),\\
-(\delta V, \gamma s V) + (\delta V^0, \gamma^0 s V^0) + \overset{\vee}{I}_\tau(\partial_t\gamma, sV\delta V) + I_\tau(s\overline{\partial}_t V, \gamma\delta\overset{\vee}{V}),
\end{aligned}$$

and estimate the inner products

$$|\ \overset{\vee}{I}_\tau(\partial_t\kappa, |\delta V|^2)| \leq \varepsilon||\delta^2 V||_Q^2 + K\,\overset{\vee}{I}_\tau(\ ||\partial_t\kappa||_{q_4}^{r_4}||\delta V||^2\ ) + Kd^2,$$

$$|(\overline{\partial}_t \delta V, P)_Q| \le \varepsilon ||\delta V||^2 + \varepsilon ||\sqrt{\tau}\overline{\partial}_t \delta V||_Q^2 + Kd^2,$$

$$|\overset{\vee}{I}_\tau(\partial_t \gamma, sV\delta V)| \le \varepsilon ||\delta^2 V||_Q + K \overset{\vee}{I}_\tau(\ ||\partial_t \gamma||_{q_8}^{r_8} ||\delta V||^2\ ) + Kd^2,$$

$$|I_\tau(s\overline{\partial}_t V, \gamma \delta \overset{\vee}{V})| \le \varepsilon ||\overline{\partial}_t V||^2 + K I_\tau ||\delta \overset{\vee}{V}||^2 + Kd^2.$$

From lemma 3.1b and obtained estimates we get the inequality

$$||\delta V||_Q^2 \le K I_\tau(\ ||\delta \overset{\vee}{V}||^2 + ||\delta \kappa||_{2q_5}^{2r_5} ||\delta V||^2)$$

$$+K \overset{\vee}{I}_\tau\big(\ (||\partial_t \kappa||_{q_4}^{r_4} + ||\partial_t \gamma||_{q_8}^{r_8}) ||\delta V||^2\ \big) + Kd^2.$$

Now we use Gronwall's lemma which proves this part of the lemma.

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